Nonlinear Dynamics and Chaos II. Assignment 6

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1 Local strong stable manifold

Given the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \mathbf{x} \in \mathbb{R}^2, \quad \mathbf{f}(\mathbf{0}) = \mathbf{0}, \quad \text{Spect}[\text{D}\mathbf{f}(\mathbf{0})] = \{-\lambda_{ss}, -\lambda_s\} \tag{1}$$

with $\lambda_{ss} > \gamma > \lambda_s > 0$.

Step 1: Coordinate change As a first step we change the coordinates by using the eigenvectors as transformation basis. Analogously to the spectral matrix decomposition, choosing these particular coordinates will expose the eigenvalues during linearization in a diagonal matrix. In our case this is straight forward since we have to distinct real eigenvalues with two related eigenvectors forming an orthogonal basis.

$$\begin{aligned} & T \coloneqq \begin{bmatrix} e_{ss} & e_s \end{bmatrix} \qquad e_{ss}, e_s \text{ eigenvectors of } \text{D}\mathbf{f}(\mathbf{0}) \\ & \mathbf{x} = T\,\mathbf{y} \\ \Rightarrow \quad & \dot{\mathbf{y}} = T^{-1}\dot{\mathbf{x}} = T^{-1}\left(\text{D}\mathbf{f}(\mathbf{0})\,\mathbf{x} + \mathcal{O}\left(|\mathbf{x}|^2\right)\right) = T^{-1}\text{D}\mathbf{f}(\mathbf{0})\,T\,\mathbf{y} + \mathcal{O}\left(|\mathbf{x}|^2\right) \end{aligned}$$

Finally we get, when introducing $g_{ss}(\mathbf{y})$ and $g_s(\mathbf{y})$ for the nonlinear terms, the following system:

$$\begin{cases} \dot{y}_{ss} = -\lambda_{ss}\, y_{ss} + g_{ss}(\mathbf{y}) \\ \dot{y}_s = -\lambda_s\, y_s + g_s(\mathbf{y}) \end{cases} \tag{2}$$

Step 2: Definition of local strong stable manifold We now define the local strong stable manifold as a set of initial conditions in a δ-region around $\mathbf{0}$, which show a decay rate of *at least* γ. In this way we make sure, that even after an eventual perturbation by non-linear terms towards a slower decay rate, we still include the perturbed manifold in our definition:

$$W^{ss}_{loc}(0) = \left\{ \mathbf{y}_0 \in B_\delta \mid \sup_{t \geq 0} e^{\gamma t} \left\| \mathbf{F}^t(\mathbf{y}_0) \right\| < \infty \right\} \tag{3}$$

It is important to note, that outside the small neighborhood δ we change the system smoothly to its linear approximation, using a C^∞ bump function. This is necessary, since the definition

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to be bounded (even though following a faster decay rate than γ), in a non linear system might generally not be unique. We are going to use an analogous notation to the one used in class:

$$\mathbf{G}_{ss,s}(\mathbf{y}) := \begin{cases} \mathbf{g}_{ss,s}(\mathbf{y}) & \text{for } \|\mathbf{y}\| < \delta \\ \leq \mathbf{g}_{ss,s}(\mathbf{y}) & \text{for } \delta < \|\mathbf{y}\| < 2\,\delta \\ 0 & \text{for } \|\mathbf{y}\| > 2\,\delta \end{cases}$$

With the bump function $\Psi_\delta(r)$, we simply have:

$$\mathbf{G}_{ss,s} = \Psi_\delta\left(\|\mathbf{y}\|\right)\, \mathbf{g}_{ss,s}(\mathbf{y})$$

To find an upper bound on the derivative of this function we make a couple of observations:

- The bump function has the property that its derivative can be upper estimated by $\frac{K_1}{\delta}$.
- The nonlinear function maybe upper estimated in this neighborhood by $K_2\,\delta^2$. Further away from the origin the overall function is identically to zero, as the bump function takes care of this.

With this knowledge we construct a *global* bound:

$$\begin{aligned} \left\|\mathbf{G}'_{s,u}\right\| &\leq \left\|\Psi'_\delta\, \mathbf{g}_{ss,s}\right\| + \left\|\Psi_\delta\, \mathbf{g}'_{ss,s}\right\| \\ &\leq \frac{K_1}{\delta}\, K_2\, \delta^2 + 1\, K_2\, \delta \leq K_3\, \delta \end{aligned} \tag{4}$$

Finally, we can rewrite our system with the bump functions as follows:

$$\begin{cases} \dot{y}_{ss} = -\lambda_{ss}\, y_{ss} + G_{ss}(\mathbf{y}) \\ \dot{y}_s = -\lambda_s\, y_s + G_s(\mathbf{y}) \end{cases} \tag{5}$$

Step 3: Equivalent integral equations Recalling the general variation of constants formula:

$$\mathbf{x}(t) = e^{A\,(t-t_0)}\, \mathbf{x}(t_0) + \int_{t_0}^{t} e^{A(t-s)}\, \mathbf{f}\left(\mathbf{x}\left(s\right), s\right) ds \tag{6}$$

we can apply this to eq. (5) using different initial times t_{ss} and t_s, since we can apply this principle independently to both equations, by always using the same full solution $\mathbf{y}(t)$ but from different t_0's. This yields:

$$\begin{cases} y_{ss}(t) = e^{-\lambda_{ss}(t-t_{ss})}\, y_{ss}(t_{ss}) + \displaystyle\int_{t_{ss}}^{t} e^{-\lambda_{ss}(t-s)} G_{ss}(\mathbf{y}(s))\, ds \\ y_s(t) = e^{-\lambda_s(t-t_s)}\, y_s(t_s) + \displaystyle\int_{t_s}^{t} e^{-\lambda_s(t-s)} G_s(\mathbf{y}(s))\, ds \end{cases} \tag{7}$$

In general we see this as a functional operator, acting on $\mathbf{y}(t)$ and the initial conditions, returning a function $\mathbf{y}(t)$ again. The actual solution is a fixed point of this operator, meaning that it returns the same function as we put in.

$$\mathbf{y}(t) = \mathcal{F}(\mathbf{y}(t), y_{ss}(t_{ss}), y_s(t_s)) \tag{8}$$

The related complete metric space on which this operates on, will be defined in the next paragraph more strictly, as things simplify by picking specific initial times.

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Step 4: Picking initial times Similar to what we have performed in class, we pick the following initial times:

$$t_s \to \infty$$
$$t_{ss} = 0$$

To further restrict the set of possible solutions, we limit ourselves to the interesting functions for which:

$$y_{ss}(0) = \delta$$

With the illustration in fig. 1 these choices become perhaps more clear.

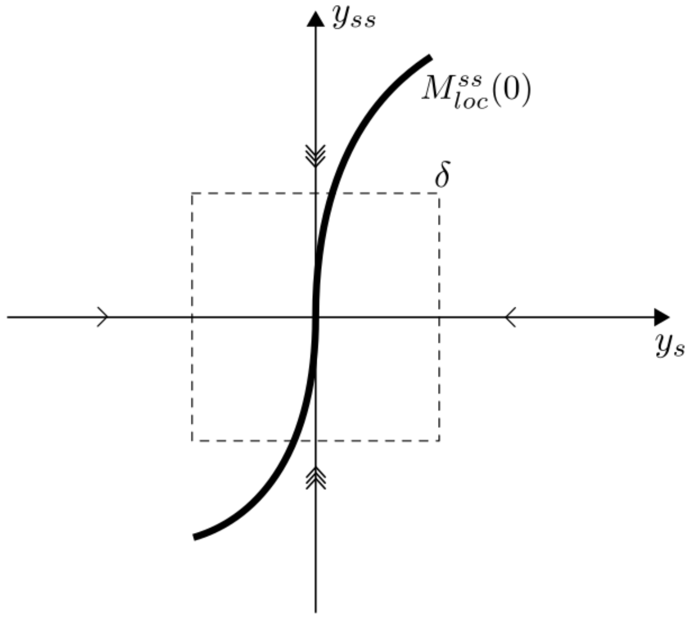

Figure 1: Illustration of strong stable manifold

Equation (7) in this way simplifies *on the local strong stable manifold* to:

$$\begin{cases} y_{ss}(t) = e^{-\lambda_{ss} t}\, y_{ss}(0) + \int_{t_{ss}}^{t} e^{-\lambda_{ss}(t-s)} G_{ss}(\mathbf{y}(s))\, ds \\ y_s(t) = \int_{\infty}^{t} e^{-\lambda_s(t-s)} G_s(\mathbf{y}(s))\, ds \end{cases} \tag{9}$$

We now properly define the Banach space of possible solutions to be:

$$\mathcal{B} = \left\{ \mathbf{y}(t) : [0, \infty) \to \mathbb{R}^2, y \in C^0, \sup_{t \geq 0} \|\mathbf{y}(t)\|\, e^{\gamma t} \leq K < \infty,\ y_{ss}(0) = \delta \right\} \tag{10}$$

We define a general notion of distance on this space of continuous functions:

$$d(a, b) := \sup_{t \geq 0} \|a - b\|\, e^{\gamma t} \tag{11}$$

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To show that the equations (9) actually map back into $\mathcal{B}$, we need to verify that after applying the mapping our notion of distance still is $\leq K$, as the definition in eq. (10) requires.

$$\begin{aligned}
\|\mathbf{y}(t)\| &\leq |y_{ss}(t)| + |y_s(t)| \\
&\leq e^{-\lambda_{ss} t}\, |y_{ss}(0)| + \int_0^t e^{-\lambda_{ss}(t-s)}\, |G_{ss}(\mathbf{y}(s))|\; ds + \int_t^\infty e^{-\lambda_s(t-s)}\, |G_s(\mathbf{y}(s))|\; ds \\
&\leq e^{-\lambda_{ss} t}\, \delta + \int_0^t e^{-\lambda_{ss}(t-s)} K_3\, \delta\, \|\mathbf{y}(s)\|\; ds + \int_t^\infty e^{-\lambda_s(t-s)} K_4\, \delta\, \|\mathbf{y}(s)\|\; ds
\end{aligned}$$

Note that we used the *global* upper estimate on the derivative from eq. (4) to upper estimate the overall function. This works for both parts G_{ss} and G_s since we have been using the same bump function.

Now we use the fact that the incoming $\mathbf{y}(s)$ on which we are evaluating the functional, is part of the Banach space in the first place. From its definition in eq. (10) we have

$$\sup_{t\geq 0} \|\mathbf{y}(t)\|\; e^{\gamma t} \leq K$$

By definition of supremum, this means that:

$$K \geq \sup_{t\geq 0} \|\mathbf{y}(t)\|\; e^{\gamma t} \geq \|\mathbf{y}(s)\|\; e^{\gamma s} \qquad \forall\, s \geq 0$$

Since $e^{\gamma s} \geq 1$ we can divide through this positive number, and finally have:

$$\|\mathbf{y}(s)\| \leq K\, e^{-\gamma s} \qquad \forall\, s \geq 0 \tag{12}$$

Proceeding with the inequality from above we have:

$$\begin{aligned}
\|\mathbf{y}(t)\| &\leq e^{-\lambda_{ss} t}\, \delta + \int_0^t e^{-\lambda_{ss}(t-s)} K_3\, \delta\, K\, e^{-\gamma s}\, ds + \int_t^\infty e^{-\lambda_s(t-s)} K_4\, \delta\, K\, e^{-\gamma s}\, ds \\
&= e^{-\lambda_{ss} t}\, \delta + K\, \delta \left[K_3\, e^{-\lambda_{ss} t} \int_0^t e^{s\,(\lambda_s s - \gamma)}\, ds + K_4\, e^{-\lambda_s t} \int_t^\infty e^{s\,(\lambda_s - \gamma)}\, ds \right] \\
&= e^{-\lambda_{ss} t}\, \delta + K\, \delta \left[\frac{K_3}{\lambda_{ss} - \gamma}\, e^{-\lambda_{ss} t}\; e^{s\,(\lambda_{ss} - \gamma)}\Big|_{s=0}^{t} + \frac{K_4}{\lambda_s - \gamma}\, e^{-\lambda_s t}\; e^{s\,(\lambda_s - \gamma)}\Big|_{s=t}^{\infty} \right] \\
&= e^{-\lambda_{ss} t}\, \delta + K\, \delta \left[\frac{K_3}{\lambda_{ss} - \gamma}\; (e^{-\gamma t} - 1) + \frac{K_4}{\gamma - \lambda_s}\, e^{-\gamma t} \right] \\
&\leq e^{-\lambda_{ss} t}\, \delta + K\, \delta \left[\frac{K_3}{\lambda_{ss} - \gamma}\, e^{-\gamma t} + \frac{K_4}{\gamma - \lambda_s}\, e^{-\gamma t} \right]
\end{aligned}$$

Note that in this calculations it is important to be aware of the assumption $\lambda_{ss} > \gamma > \lambda_s$So to summarize we end up with:

$$\|\mathbf{y}(t)\| \leq e^{-\lambda_{ss} t}\, \delta + K\, \delta\, e^{-\gamma t} \left(\frac{K_3}{\lambda_{ss} - \gamma} + \frac{K_4}{\gamma - \lambda_s} \right)$$

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By multiplying this inequality with the strictly positive number $e^{\gamma t}$, we can pass from the point-wise euclidean norm, to an expression similar to our notion of distance in the defined Banach space:

$$\begin{aligned}\|\mathbf{y}(t)\|\ e^{\gamma t} &\leq e^{-(\lambda_{ss}-\gamma)\,t}\,\delta + K\,\delta\left(\frac{K_3}{\lambda_{ss}-\gamma}+\frac{K_4}{\gamma-\lambda_s}\right)\\ &\leq \delta + K\,\delta\left(\frac{K_3}{\lambda_{ss}-\gamma}+\frac{K_4}{\gamma-\lambda_s}\right)\end{aligned}$$

Note that the right hand side *does not depend on time*. This means that taking the $\sup_{t\geq 0}$ will not change this upper bound. Furthermore, the right hand side scales linearly with δ, which means that we will always be able to choose a δ, such that the overall expression $< K$, i.e. we do not leave our previously defined Banach space.

Therefore we can finally define the proper mapping to be:

$$\mathcal{F}(\mathbf{y}(t)) : \mathcal{B} \to \mathcal{B} \qquad \Longleftrightarrow \qquad \sup_{t\geq 0}\|\mathcal{F}(\mathbf{y}(t))\|\,e^{\gamma t} \leq K \qquad \text{for } \delta \text{ small enough} \tag{13}$$

Step 5: Contraction To show that our mapping is indeed a contraction on its Banach space, we will use our notion of distance defined in eq. (11).

$$d\left(\mathcal{F}\left(\mathbf{y}^1\right),\mathcal{F}\left(\mathbf{y}^2\right)\right) = \sup_{t\geq 0}\left\|\mathcal{F}\left(\mathbf{y}^1(t)\right)-\mathcal{F}\left(\mathbf{y}^2(t)\right)\right\| e^{\gamma t}$$

Note that the initial conditions cancel out, since on the manifold (our Banach space) both are equal to δ. We develop the norm as follows:

$$\begin{aligned}\left\|\mathcal{F}\left(\mathbf{y}^1(t)\right)-\mathcal{F}\left(\mathbf{y}^2(t)\right)\right\|\,e^{\gamma t} &\leq \int_0^t e^{-\lambda_{ss}(t-s)}\left|G_{ss}(\mathbf{y}^1(s))-G_{ss}(\mathbf{y}^2(s))\right|\,e^{\gamma t}\,ds\\ &\quad+\int_\infty^t e^{-\lambda_s(t-s)}\left|G_s(\mathbf{y}^1(s))-G_s(\mathbf{y}^2(s))\right|\,e^{\gamma t}\,ds\\ &\leq \int_0^t e^{-\lambda_{ss}(t-s)}\,K_3\,\delta\,\left\|\mathbf{y}^1(s))-\mathbf{y}^2(s))\right\|\,e^{\gamma t}\,ds\\ &\quad+\int_\infty^t e^{-\lambda_s(t-s)}\,K_4\,\delta\,\left\|\mathbf{y}^1(s))-\mathbf{y}^2(s)\right\|\,e^{\gamma t}\,ds\\ &= \int_0^t e^{-\lambda_{ss}(t-s)}\,K_3\,\delta\,\left\|\mathbf{y}^1(s))-\mathbf{y}^2(s))\right\|\,e^{\gamma s}\,e^{\gamma(t-s)}\,ds\\ &\quad+\int_\infty^t e^{-\lambda_s(t-s)}\,K_4\,\delta\,\left\|\mathbf{y}^1(s))-\mathbf{y}^2(s)\right\|\,e^{\gamma s}\,e^{\gamma(t-s)}\,ds\\ &\leq K_5\,\delta\left[\sup_{t\geq 0}\left\|\mathbf{y}^1(t))-\mathbf{y}^2(t))\right\|\,e^{\gamma t}\right]\left(\int_0^t e^{(-\lambda_{ss}+\gamma)(t-s)}\,ds+\int_\infty^t e^{(-\lambda_s+\gamma)(t-s)}\,ds\right)\\ &= K_5\,\delta\,d\left(\mathbf{y}^1,\mathbf{y}^2\right)\left(\frac{1}{\lambda_{ss}-\gamma}\,e^{(\lambda_{ss}-\gamma)(s-t)}\Big|_0^t+\frac{1}{\lambda_s-\gamma}\,e^{(\lambda_s-\gamma)(s-t)}\Big|_\infty^t\right)\\ &\leq K_5\,\delta\,d\left(\mathbf{y}^1,\mathbf{y}^2\right)\left(\frac{1}{\lambda_{ss}-\gamma}+\frac{1}{\lambda_s-\gamma}\right)\end{aligned}$$

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Note that $K_5 = \max\{K_3, K_4\}$. Similarly to the case of before, we have reached a time independent right hand side, as an upper estimate. When taking the $\sup_{t\geq 0}$ on the left hand side, the upper estimate does therefore still hold, and we finally have, by summarizing the constant terms on the right hand side with K_6:

$$d\left(\mathcal{F}\left(\mathbf{y}^1\right), \mathcal{F}\left(\mathbf{y}^2\right)\right) \leq K_6\, \delta\, d\left(\mathbf{y}^1, \mathbf{y}^2\right) \tag{14}$$

This means, when keeping δ small enough, we can guarantee a contraction factor < 1.

$$\Rightarrow \exists!\ W^{ss}_{loc} \text{ in } B_\delta$$

□ *q.e.d.*

2 Spectral Submanifold

Consider the system

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -\sqrt{14}\, y + x^2 + x^3 + x^4 \end{cases} \tag{15}$$

It is easy to verify that the linear part of the system has two eigenvalues:

$$\lambda_1 = -1, \quad \lambda_2 = -\sqrt{14}$$

The interesting spectral subspace to choose is hereby the x-direction, since the y-direction is fast decaying and will have a unique strong stable manifold. After a fast decay in y-direction the dynamics on the x-direction will be more valuable from an engineering point of view, as this might be the direction along which one can perhaps control the system. Formally we define:

$$E = \left\{ x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\}, \quad \dim E = 1 \tag{16}$$

The spectral quotient of E is then readily computed by:

$$\sigma(E) = \mathrm{Int}\left[\frac{-\sqrt{14}}{-1}\right] = 3 \tag{17}$$

2.1 Existence of analytic slow SSM

First we verify the low order non-resonance condition:

$$\begin{aligned} m\, \lambda_1 &\neq \lambda_2 \\ m\,(-1) &\neq -\sqrt{14} \qquad \forall\, m \in \mathbb{N} \,:\, 2 \leq m \leq 3 \end{aligned}$$

which is satisfied. We further observe that the right hand side of our system in eq. (15) is C^a in (x, y). Therefore we conclude with the SSM theorem that:

(i) $\exists C^a$ SSM, $W_E(0)$ tangent to E at $(x, y) = \mathbf{0}$

(ii) $W_E(0)$ is *unique* already among $C^{\sigma(E)+1} = C^4$ invariant manifolds satisfying (i)

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2.2 Expression for SSM

We construct the expression for the manifold as a *graph* over the x-axis:

$$y = h(x) = a\,x + b\,x^2 + c\,x^3 + d\,x^4 + \mathcal{O}(x^5) \tag{18}$$

Note that given the results in section 2.1 we guarantee by using these 4 terms, that we are able to identify the unique smoothest SSM.

We now can express $\dot{y}$ in two different ways, given the invariance of the manifold:

$$\begin{aligned}\dot{y} &= h'(x)\,\dot{x} = -h'(x)\,x = -a\,x - 2\,b\,x^2 - 3\,c\,x^3 - 4\,d\,x^4 \\ &= -\sqrt{14}\,h(x) + x^2 + x^3 + x^4 = -\sqrt{14}\,a\,x + (1-\sqrt{14}\,b)\,x^2 + (1-\sqrt{14}\,c)\,x^3 + (1-\sqrt{14}\,d)\,x^4\end{aligned}$$

By comparing the coefficients we get the system:

$$\begin{cases}-\sqrt{14}\,a = -a \\ 1-\sqrt{14}\,b = -2\,b \\ 1-\sqrt{14}\,c = -3\,c \\ 1-\sqrt{14}\,d = -4\,d\end{cases} \quad\Rightarrow\quad \begin{cases} a = 0 \\ b = \frac{1}{\sqrt{14}-2} \\ c = \frac{1}{\sqrt{14}-3} \\ d = \frac{1}{\sqrt{14}-4}\end{cases} \tag{19}$$

It easy to verify that further expansion would lead to zero order terms, i.e.: $-\sqrt{14}\,e = -5\,e \quad\Rightarrow\quad e = 0$. Therefore we can can conclude, that:

$$h(x) = \frac{1}{\sqrt{14}-2}\,x^2 + \frac{1}{\sqrt{14}-3}\,x^3 + \frac{1}{\sqrt{14}-4}\,x^4 \tag{20}$$

is an *exact* expression for the slow smoothest SSM for the chosen E.

2.3 General solution of the dynamical system

The dynamics on the x-axis is a decoupled dynamical system, and a general solution with $x(0) = x_0$ is given by:

$$x(t) = x_0\,e^{-t} \tag{21}$$

Using this general solution in the y dynamics, converts them to an inhomogeneous first order linear ODE:

$$\dot{y} + \sqrt{14}\,y = x_0^2\,e^{-2\,t} + x_0^3\,e^{-3\,t} + x_0^4\,e^{-4\,t} \tag{22}$$

The so-called variation of constant method yields:

$$\begin{aligned} y(t) &= e^{-\sqrt{14}\,t}\,y_0 + \int_0^t e^{-\sqrt{14}(t-s)}(x_0^2\,e^{-2\,s} + x_0^3\,e^{-3\,s} + x_0^4\,e^{-4\,s})ds \\ &= e^{-\sqrt{14}\,t}\,y_0 + \int_0^t \left(x_0^2\,e^{(\sqrt{14}-2)s-\sqrt{14}\,t} + x_0^3\,e^{(\sqrt{14}-3)s-\sqrt{14}\,t} + x_0^4\,e^{(\sqrt{14}-4)s-\sqrt{14}\,t}\right)ds \\ &= e^{-\sqrt{14}\,t}\,y_0 + \frac{x_0^2}{\sqrt{14}-2}\left(e^{-2\,t} - e^{-\sqrt{14}t}\right) + \frac{x_0^3}{\sqrt{14}-3}\left(e^{-3\,t} - e^{-\sqrt{14}t}\right) + \frac{x_0^4}{\sqrt{14}-4}\left(e^{-4\,t} - e^{-\sqrt{14}t}\right)\end{aligned}$$

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A quick check confirms that $y(0) = y_0$, as the x_0 related terms cancel out. To get manifold candidates we eliminate time, by substituting $e^{-t} \to \frac{x}{x_0}$ for $x_0 \neq 0$. This yields after some simplifications:

$$y(x) = y_0 \left(\frac{x}{x_0}\right)^{\sqrt{14}} + \frac{x^2 - x_0^2 \left(\frac{x}{x_0}\right)^{\sqrt{14}}}{\sqrt{14}-2} + \frac{x^3 - x_0^3 \left(\frac{x}{x_0}\right)^{\sqrt{14}}}{\sqrt{14}-3} + \frac{x^4 - x_0^4 \left(\frac{x}{x_0}\right)^{\sqrt{14}}}{\sqrt{14}-4} \qquad x_0 \neq 0 \tag{23}$$

Regarding differentiability, the problematic term is $x^{\sqrt{14}}$. After differentiating 3 times, the derivative in $x = 0$ will not be defined anymore. These functions are therefore at most C^3. By wisely picking y_0 we can actually cancel out those terms:

$$\text{choose} \quad y_0 = \frac{1}{\sqrt{14}-2} x_0^2 + \frac{1}{\sqrt{14}-3} x_0^3 + \frac{1}{\sqrt{14}-4} x_0^4$$
$$\Rightarrow y(x) = \frac{1}{\sqrt{14}-2} x^2 + \frac{1}{\sqrt{14}-3} x^3 + \frac{1}{\sqrt{14}-4} x^4$$

Hereby we show indeed that this particular choice, of y_0 in relation to x_0, yields *uniquely* the most smooth manifold, being C^a in x.

2.4 Plots

The negative coefficient of x^4 in eq. (20) dominates the manifold soon outside the neighborhood of $x = 0$ and therefore dimensions grow quickly. Figure 2 shows therefore a plot with large dimensions on the y-axis.

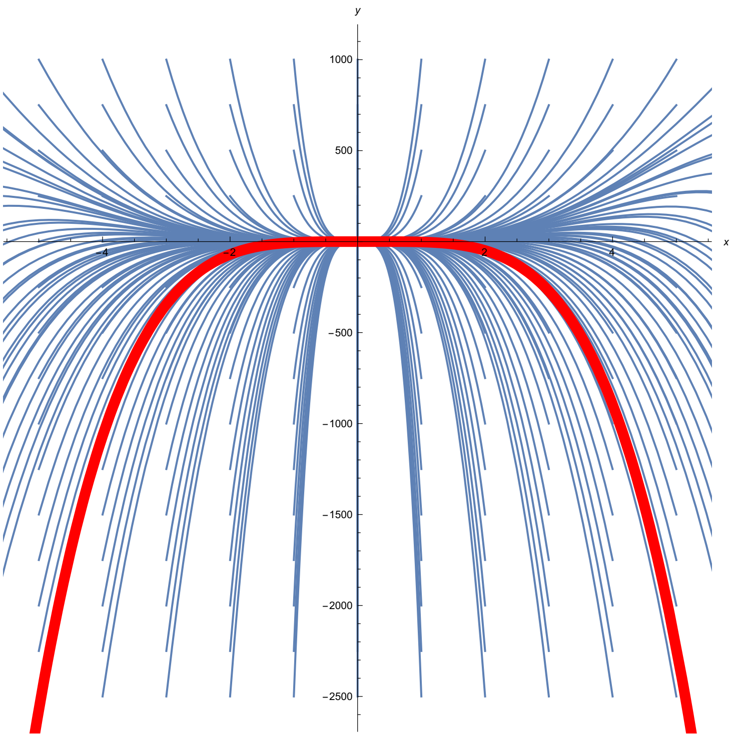

Figure 2: Trajectories approaching origin with the SSM

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Figure 3 shows a close up on the region $x \in [-2, 2]$, where the manifold is still near the subspace E.

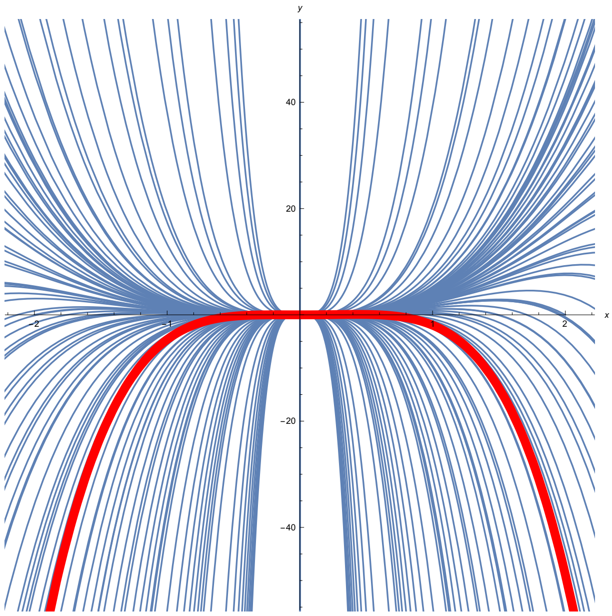

Figure 3: Closeup around the origin

It is interesting to realize that in very close vicinity the trajectories tempt to "overshoot" the x-axis, and the manifold somehow imitates this by being $y > 0$ in some parts near the origin. An illustration is provided in fig. 4 However, as the large scale plots in figs. 2 and 3 show, these details are negligible with respect to the overall behavior.

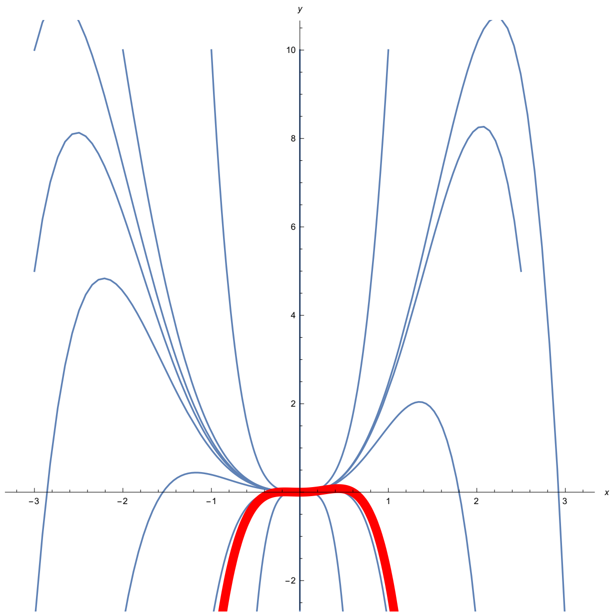

Figure 4: Trajectories around the origin

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2.5 Reduced order model

The exact reduced order model is simply given by

$$\dot{x} = -x \tag{24}$$

Note that the expression for the manifold does not influence the dynamics, since in the subspace E these are decoupled from the system, i.e. no dependence on y is observed.

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