Nonlinear Dynamics and Chaos II. Assignment 5

Florian Mahlknecht

2020-05-13

1 Slowly forced pendulum

Given the slowly forced pendulum equation:

$$\ddot{\varphi} + k\,\dot{\varphi} + \sin\varphi = F_0\,\sin(\varepsilon\,t) \qquad k > 0,\; 0 < F_0 < 1,\; 0 < \varepsilon \ll 1 \tag{1}$$

we can rewrite the equation as a 3-dimensional dynamical system, using the following variables:

$$\begin{bmatrix} x_1 \\ x_2 \\ \psi \end{bmatrix} = \begin{bmatrix} \varphi \\ \dot{\varphi} \\ \varepsilon\,t \end{bmatrix} \tag{2}$$

This yields:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k\,x_2 - \sin x_1 + F_0\,\sin\psi \\ \dot{\psi} = \varepsilon \end{cases} \qquad \psi(0) = 0 \tag{3}$$

In the limit case of 0 forcing frequency, i.e. $\varepsilon = 0$, the system reduces to the pendulum with viscous damping, i.e. has an asymptotically stable fixed point in the origin. More generally we can define critical manifolds, satisfying the conditions

$$x_2 = 0,\; \sin x_1 = F_0\,\sin\psi,\; \psi \in \mathbb{R}$$
$$\Rightarrow \qquad x_1 = \begin{cases} \arcsin(F_0\,\sin\psi) \\ \pi - \arcsin(F_0\,\sin\psi) \end{cases}$$

Note that due to the fact that $\sin(x) = \sin(\pi - x)$, the complete solution projects *two* points onto the x-axis in the 2π interval $[-\frac{\pi}{2}, \frac{3\pi}{2}]$ for a single value of ψ. Due to the periodic nature of the positional angle coordinate x_1 this extends to additive multiples of 2π. To account for this fact, we will think of $x_1 \in \mathbb{S}_1$, and include this periodicity in our phase portrait.

Note that one manifold will be the extension of the asymptotically stable fixed point $x_1 = 0$, whereas the other will be the extension of the unstable fixed point $x_1 = \pi$.

Figure 1 tries to illustrate this concept in the extended periodic phase space.

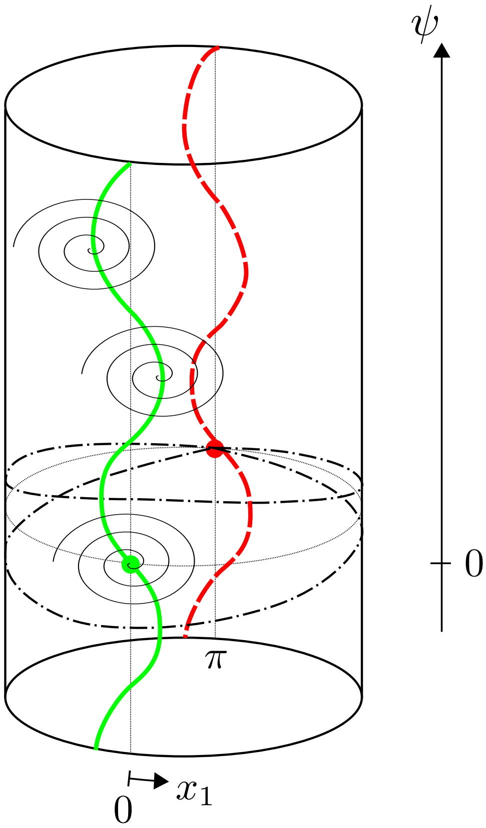

Figure 1: Slowly forced pendulum phase space

Note that the x_2 axis is not explicitly shown, for reason of illustration and imagination however, some parts of trajectories and the separatrix of the undamped pendulum is shown along the same axis as ψ.

The green manifold in the front is how the asymptotically stable fixed point *slowly* varies along time ψ. By slow variation of the forcing along the axis ψ, we can see that this fixed point starts to move. Trajectories will then converge to this displaced fixed point, when the variation is slow enough. Imagine to be already on the stable fixed point, then intuitively we would just simply follow the forcing term, which gives the *invariance* of the stable manifold.

Regarding the red unstable manifold in the back, it is perhaps interesting to note the sign change:

$$x_1 = \pi - \arcsin(F_0 \sin\psi) \tag{4}$$

This means that physically we move the stable fixed point, for instance, to the left, and the unstable fixed points moves to the left on top of the pendulum as well. This again intuitively makes sense, when we imagine to attach an actuator with a clock-wise turning torque to the shaft of the pendulum. Note that the invariance in this case is most likely not given, since all trajectories clearly diverge from this edge case. From a physical point of view, even in the unlikely case of hitting exactly this limit perturbation, the start of a clockwise torque, would certainly lead to divergence. A rigorous mathematical discussion is out of scope for this qualitative analysis provided in here. It is important to emphasize that the term “manifold” in this regard is used as an intuitive geometric concept rather than mathematically rigorously established.

Florian Mahlknecht
fmahlknecht@student.ethz.ch
19-945-351

2 Thermohaline Criculation

Consider the system

$$\begin{cases} \dot{x} = -\frac{1}{\tau_x}(x-1) + \frac{1}{\tau_y} x \left[1+\eta^2 (x-y)^2\right] \\ \dot{y} = -\frac{\mu}{\tau_z} - \frac{1}{\tau_y} y \left[1+\eta^2 (x-y)^2\right] \end{cases} \qquad \frac{\tau_x}{\tau_y} = \varepsilon \ll 1 \tag{5}$$

2.1 Slow Manifold

If we first rescale time using

$$s := \frac{t}{\tau_y}$$

we have

$$\begin{aligned} \frac{d(\bullet)}{dt} &= \frac{d(\bullet)}{ds}\frac{ds}{dt} \\ \frac{ds}{dt} &= \frac{1}{\tau_y} \\ \Rightarrow \frac{d(\bullet)}{dt} &= \frac{1}{\tau_y}\frac{d(\bullet)}{ds} \end{aligned}$$

Applying this to eq. (5), while multiplying by τ_y, yields:

$$\begin{cases} \frac{dx}{ds} = -\frac{\tau_y}{\tau_x}(x-1) + x\left[1+\eta^2 (x-y)^2\right] \\ \frac{dy}{ds} = \mu - y\left[1+\eta^2 (x-y)^2\right] \end{cases} \tag{6}$$

Using the ε notion $\frac{\tau_x}{\tau_y} = \varepsilon \ll 1$, we can rewrite the system as a regular singular perturbation problem:

$$\begin{cases} \varepsilon\frac{dx}{ds} = -\frac{\tau_y}{\tau_x}(x-1) + \varepsilon x\left[1+\eta^2 (x-y)^2\right] \\ \frac{dy}{ds} = \mu - y\left[1+\eta^2 (x-y)^2\right] \end{cases} \tag{7}$$

As Fenichel proposed, we can introduce a *fast* time, and turn it into a regular perturbation problem now:

$$\tau := \frac{1}{\varepsilon}s$$

For convenience we now introduce a new symbol for deriving in τ, getting similarly to the steps performed before:

fmahlknecht@student.ethz.ch
19-945-351

$$(\bullet)' \coloneqq \frac{d(\bullet)}{d\tau} = \frac{1}{\varepsilon}\frac{d(\bullet)}{d\tau}$$

This allows us to rewrite the system as follows:

$$\begin{cases} x' = -(x-1) + \varepsilon x\left[1 + \eta^2 (x-y)^2\right] \\ y' = \varepsilon\left[\mu - y\left(1 + \eta^2 (x-y)^2\right)\right] \end{cases} \tag{8}$$

In the physically meaningless $\varepsilon = 0$ limit, where time stands still, we can find a manifold of fixed points:

$$\begin{cases} x' = -(x-1) \\ y' = 0 \end{cases}$$

The so called slow manifold can then be defined as:

$$M_0 = \{x = 1, y \in \mathbb{R}\} \tag{9}$$

Figure 2 shows a small sketch of the unperturbed manifold.

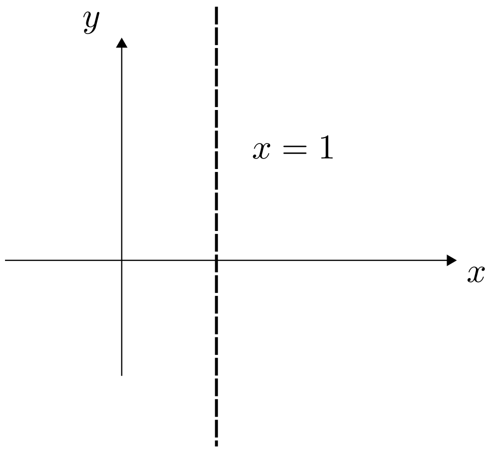

Figure 2: M_0 sketch

Stability Similarly to the examples solved in class we can define

$$\varphi_0(y) = 1$$
$$\eta \coloneqq x - \varphi_0(y)$$

i.e. we express the manifold as a graph over the y-axis, and introduce a new variable η, which simply flattens out this graph, although in our case φ_0 is constant. In this way the stability of $\eta(y)$ around 0, for all y, unveils the dynamical behavior around the manifold. Deriving in τ yields:

$$\begin{aligned} \eta' &= x' - \mathrm{D}\varphi_0(y)\, y' = x' - 0 \\ &= -(x-1) = -(\varphi_0(y) + \eta - 1) = -(1 + \eta - 1) \\ &= -\eta \end{aligned}$$

Florian Mahlknecht
fmahlknecht@student.ethz.ch
19-945-351

This means that independently of the position y on the manifold, i.e. for $\eta = 0$ the system is asymptotically stable (eigenvalue -1). In other words the manifold is globally attracting.

□ $q.e.d.$

Leading order approximation For $\varepsilon > 0$ we Taylor expand:

$$x = \varphi_\varepsilon(y) = \varphi_0(y) + \varepsilon\,\varphi_1(y) + \frac{1}{2}\varepsilon^2\varphi_2(y) + \mathcal{O}(\varepsilon^3) \tag{10}$$

The invariance of the manifold, guarantees that *on the manifold* we have:

$$x' = \mathrm{D}\varphi_\varepsilon(y)\,y' = f(x, y, \varepsilon)\big|_{x=\varphi_\varepsilon(y)}$$

Where in this equation $f(x, y, \varepsilon)$ indicates the right hand side of the x evolution in our system. Considering first terms up to $\mathcal{O}(\varepsilon)$, and using $\varphi_0(y) = 0$, we get:

$$\left(\varepsilon\,\frac{d\,\varphi_1}{dy} + \mathcal{O}(\varepsilon^2)\right)\varepsilon\left[\mu - y\left(1 + \eta^2\left(1 + \varepsilon\,\varphi_1(y) + \mathcal{O}(\varepsilon^2) - y\right)^2\right)\right] =$$
$$-\left(1 + \varepsilon\,\varphi_1(y) + \mathcal{O}(\varepsilon^2) - 1\right) + \varepsilon\,\left(1 + \varepsilon\,\varphi_1(y) + \mathcal{O}(\varepsilon^2)\right)\left[1 + \eta^2\left(1 + \varepsilon\,\varphi_1(y) + \mathcal{O}(\varepsilon^2) - y\right)^2\right]$$

Which simplifies to:

$$\mathcal{O}(\varepsilon^2) = \varepsilon\left[-\varphi_1(y) + 1 + \eta^2\,(1 - y)^2\right] + \mathcal{O}(\varepsilon^2)$$

This unveils:

$$\varphi_1(y) = 1 + \eta^2\,(1 - y)^2$$

And therefore the leading order approximation becomes:

$$x = \varphi_\varepsilon(y) = 1 + \varepsilon\,\left(1 + \eta^2\,(1 - y)^2\right) + \mathcal{O}(\varepsilon^2) \tag{11}$$

Figure 3 shows how the perturbed manifold might look like.

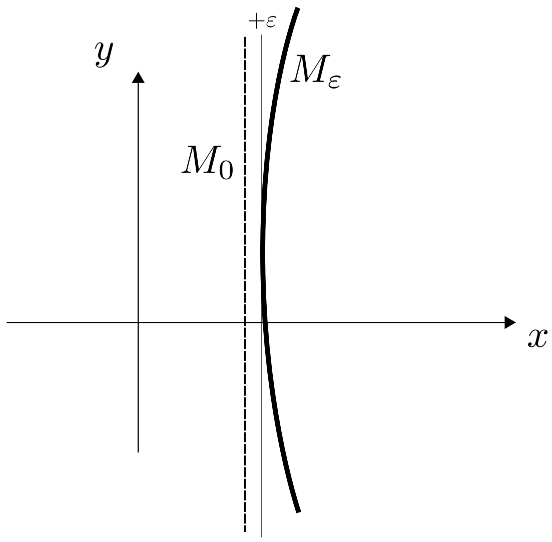

Figure 3: M_ε sketch

fmahlknecht@student.ethz.ch
19-945-351

2.2 Reduced flow

Given the globally attracting slow manifold from the previous section we can now restrict the dynamics to the manifold, in order to get insight of how trajectories might behave in general, since given the global attraction it is likely to observe them. The flow on the manifold is uniquely determined by the evolution of y, since on the manifold $x = \varphi_\varepsilon(y)$. Therefore:

$$\begin{aligned} y' &= \varepsilon\left[\mu - y\left(1 + \eta^2(x-y)^2\right)\right]\Big|_{x=\varphi_\varepsilon(y)} \\ &= \varepsilon\left[\mu - y\left(1 + \eta^2\left(1 + \varepsilon\left(1 + \eta^2(1-y)^2\right) + \mathcal{O}(\varepsilon^2) - y\right)^2\right)\right] \\ &= \varepsilon\left[\mu - y\left(1 + \eta^2(1-y)^2\right)\right] + \mathcal{O}(\varepsilon^2) \end{aligned}$$

The reduced flow is hence governed by the equation:

$$y' = \varepsilon\left[\mu - y + y\,\eta^2(1-y)^2\right] + \mathcal{O}(\varepsilon^2) \tag{12}$$

In $y = 0$ we will have a positive or negative evolution depending on the sign of μ. Zero coupling between salinity and temperature, i.e. $\mu = 0$ would mean having a fixed point there.

For $\eta \neq 0$, we have a 3rd order equation in y, which for large enough y is $\gg 0$, and for $y \ll 0$, $y \ll 0$. This means that for $|y| \gg 0$ the we will leave the manifold. If we restrict it in a suitable way, we would therefore get an *overflowing* invariant manifold.

Given that it is a third order equation in y, we expect to see two local extrema:

$$\begin{aligned} g(y) &= \varepsilon\left[\mu - y + y\,\eta^2(1-y)^2\right] \\ \frac{dg(y)}{dy} &= \varepsilon\left[-\eta^2 y^2 \eta^2 - 1\right] = 0 \\ &\iff 3\eta^2 y^2 - 4\eta^2 y + (\eta^2 - 1) = 0 \end{aligned}$$

The roots of this polynomial are given by

$$y_{1/2} = \frac{2}{3} \pm \sqrt{\frac{1}{9} + \frac{1}{3\eta^2}} \tag{13}$$

So depending on the value of η, the couple of roots will be symmetrically around $\frac{2}{3}$. The interesting discussion is now about sign of the function value of $g(y)$ at these two extrema. Other than the value of η, the value of μ can influence this result. There is only one interesting case where the sign of both $g(y_{1/2})$ is opposite, i.e. $g(y_1) > 0$ and $g(y_2) < 0$. In this way we have 3 zero crossings with the middle one being a *stable fixed point*. In all other cases we will observe just one or two fixed points, which are both unstable, i.e. $y' > 0$ or $y' < 0$ in the vicinity of $y' = 0$. This means a slight positive or negative perturbation from these points will lead to overflow on the manifold, either $\to -\infty$ or $\to +\infty$.

 Florian Mahlknecht
fmahlknecht@student.ethz.ch
19-945-351

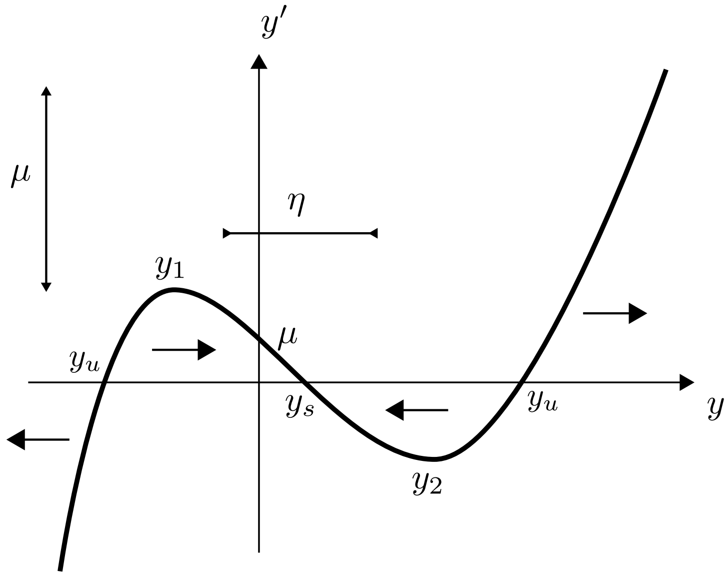

Figure 4: Reduced flow

A sketch of the most interesting case is shown in fig. 4, with an indication of the impact that changing the values μ and η has.

Varying μ will translate the whole graph $g(y)$ up and down. η will instead scale the vicinity of the two extrema y_1 and y_2. The existence of stable and unstable fixed points y_u and y_s depends on those two parameters as discussed before.

fmahlknecht@student.ethz.ch
19-945-351