# Nonlinear Dynamics and Chaos II.
# Assignment 4

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## 1 Lyapunov type numbers example

Given the system

$$\begin{cases} \dot{x} = -x\,(1-x^2) \\ \dot{y} = -b\,y \end{cases} \tag{1}$$

to calculate the Lyapunov type numbers with the linear operators, we first need to get the flow map. Note that the equations are perfectly decoupled, so we may first focus on the first one. By slightly rewriting the equations we bring it into the form of a Bernoulli differential equation with constant coefficients. A substitution with $z \coloneqq \frac{1}{x^2}$ then leads back to an ordinary differential equation:

$$\begin{aligned} \dot{x} &= -x + x^3 \\ z &\coloneqq \frac{1}{x^2} \qquad\qquad x(t) \neq 0 \quad \forall\, t \in \mathbb{R} \\ \Rightarrow \dot{z} &= 2\,z - 2 \end{aligned}$$

This first ordinary differential equation is most generally solved by:

$$z(t) = C_0\,e^{2t} + 1$$

Back-substitution then yields:

$$x(t) = \pm\frac{1}{\sqrt{C_0\,e^{2t} + 1}}$$

To define the constant $C_0$, we calculate:

$$\begin{aligned} x(0) = x_0 &= \pm\frac{1}{\sqrt{C_0 + 1}} \\ C_0 &= \frac{1}{x_0^2} - 1 \end{aligned}$$

We may observe that, by continuity of the solutions and positivity of the denominator of the obtained solution, we can use sgn $(x_0)$ to decide whether to take the positive or negative solution:

$$x(t) = \frac{\operatorname{sgn}(x_0)}{\sqrt{e^{2t}\left(\frac{1}{x_0^2} - 1\right) + 1}} \qquad\qquad x_0 \neq 0 \tag{2}$$

Note that for $x_0 = 0$ we stay in the fixed point $x = 0$. Given that close enough to the fixed point solutions are attracted to the fixed point, since the nonlinear term is too small to change sign, we can expect this part of the flow map to be continuous in $x_0$, even though we need to define it in a piecewise manner.

The second equation is straight forward solved by:

$$y(t) = y_0\, e^{-bt} \tag{3}$$

The differential of the flow map in $\mathbf{p} = (x_0, y_0)$ is hence:

$$\mathrm{D}F^t(\mathbf{p}) = \begin{bmatrix} \operatorname{sgn}(x_0)\ (e^{2t}\ (x_0^{-2} - 1) + 1)^{-\frac{3}{2}}\ \frac{e^{2t}}{x_0^3} & 0 \\ 0 & e^{-bt} \end{bmatrix} \tag{4}$$

it is important to note that the sgn function in eq. (2) is just a shorthand way to write the flow map and does not need to be derived (since this would bring complications in $x_0 = 0$). As indeed we will show later, the limit for $x_0 \to 0$ in this differential exists, and is the true derivative in the fixed point.

Given the overflowing invariant manifold

$$M_0 = \left\{(x, y) \in \mathbb{R}\ :\ y = 0,\ x \in \left[-\frac{3}{2}, \frac{3}{2}\right]\right\} \tag{5}$$

Note that the tangent space is along the $x$ direction for every position on the manifold and the normal space is along $y$. We can express this with the tangent and the normal bundles:

$$\begin{aligned} \mathrm{T}M_0 &= M_0 \times (\mathbb{R}, 0) \\ \mathrm{N}M_0 &= M_0 \times (0, \mathbb{R}) \end{aligned}$$

the projection to the normal space is therefore simply:

$$\Pi^N = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{6}$$

We get the first linear operator, by restricting the differential of the flow map, to the tangent space:

$$A_t(\mathbf{p}) = \mathrm{D}F^{-t}(\mathbf{p})\big|_{\mathrm{T}_pM} = \begin{bmatrix} \operatorname{sgn}(x_0)\ (e^{-2t}\ (x_0^{-2} - 1) + 1)^{-\frac{3}{2}}\ \frac{e^{-2t}}{x_0^3} & 0 \\ 0 & 0 \end{bmatrix} \tag{7}$$

For the second linear operator we have to project back into the normal space the differential flow map in forward time in the location mapped in backward time. It is useful to observe that

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the projection of the differential flow map, does not depend on $y_0$, so there is no need to calculate the flow of any point in backward time.

$$B_t(\mathbf{p}) = \Pi_{\mathbf{p}}^N \, DF^t\left(F^{-t}(\mathbf{p})\right) = \begin{bmatrix} 0 & 0 \\ 0 & e^{-bt} \end{bmatrix} \tag{8}$$

The matrix norm of these diagonal matrices, given that they only have one non-zero eigenvalue, is simply the absolute value of the non-zero element. In other words, a unit vector applied to these matrices can at best return the non zero element, i.e. applying $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

With this observation in mind we can now calculate the Lyapunov type numbers:

$$\nu(\mathbf{p}) = \limsup_{t\to\infty} \|B_t(\mathbf{p})\|^{\frac{1}{t}} = \limsup_{t\to\infty} \left|e^{-bt}\right|^{\frac{1}{t}} = e^{-b} \tag{9}$$

We can see that this independent from the position $\mathbf{p}$ on the manifold and for $b > 0$ it is strictly less than one.

Therefore we may calculate $\sigma(\mathbf{p})$ as well using the operators as:

$$\sigma(\mathbf{p}) = \limsup_{t\to\infty} \frac{\log \|A_t(\mathbf{p})\|}{-\log \|B_t(\mathbf{p})\|} = \limsup_{t\to\infty} \frac{\log\left|\operatorname{sgn}(x_0)\ \left(e^{-2t}\ (x_0^{-2} - 1) + 1\right)^{-\frac{3}{2}} \frac{e^{-2t}}{x_0^3}\right|}{-\log\left|e^{-bt}\right|} \tag{10}$$

For $x_0 \neq 0$ we can see that the term develop the denominator as follows:

$$\begin{aligned}
&\log\left|\operatorname{sgn}(x_0)\ \left(e^{-2t}\ \left(x_0^{-2} - 1\right) + 1\right)^{-\frac{3}{2}} \frac{e^{-2t}}{x_0^3}\right| = \\
&\log\left|\operatorname{sgn}(x_0)\right| + \log\left|\left(e^{-2t}\ \left(x_0^{-2} - 1\right) + 1\right)^{-\frac{3}{2}}\right| + \log\left|e^{-2t}\right| - \log\left|x_0^3\right| = \\
&\log\left|\left(e^{-2t}\ \left(x_0^{-2} - 1\right) + 1\right)^{-\frac{3}{2}}\right| - 2t - \log\left|x_0^3\right| = \\
&\qquad = -2t + o(t) \qquad\qquad \text{for } t \to \infty
\end{aligned}$$

For $t \to \infty$, we can clearly see that only the term $-2t$ is relevant, as the other terms remain constant or $\to 0$, in other words are $o(t)$.

Using this knowledge in eq. (10), we get:

$$\sigma(\mathbf{p}) = \limsup_{t\to\infty} \frac{-2t}{bt} = -\frac{2}{b} \qquad \forall\ \mathbf{p} = (x_0, y_0) \in M_0 : x_0 \neq 0 \tag{11}$$

To evaluate the case for $x_0 = 0$, we calculate the derivative of the (forward time) differential flow map of the interesting, $x_0$ - dependent part, by applying the definition of derivative in $x_0 = 0$, recalling that the piecewise defined flow map evaluates to the fixed point $x = 0$ in that point:

$$\left.\frac{\partial}{\partial x_0} \frac{\operatorname{sgn}(x_0)}{\sqrt{e^{2t}\left(\frac{1}{x_0^2} - 1\right) + 1}}\right|_0 = \lim_{x_0\to 0} \frac{\operatorname{sgn}(x_0)}{x_0\sqrt{e^{2t}\left(\frac{1}{x_0^2} - 1\right) + 1}} = \lim_{x_0\to 0} \frac{\operatorname{sgn}(x_0)}{x_0\sqrt{\frac{e^{2t}}{x_0^2} + \mathcal{O}(x_0^{-2})}} = \lim_{x_0\to 0} \frac{\operatorname{sgn}(x_0)}{\frac{x_0}{|x_0|}\, e^t} = e^{-t}$$

This changes the operator $A_t(\mathbf{p})$ for $x_0 = 0$ accordingly in backward time. We get:

$$\sigma\left(\mathbf{p} = (0,0)\right) = \limsup_{t\to\infty} \frac{t}{bt} = \frac{1}{b} \tag{12}$$


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# 2 Invariant Foliation

Considering the system:

$$\begin{cases} \dot{x} = -\varepsilon(x + y^2) \\ \dot{y} = -y \\ \dot{z} = z \end{cases} \qquad \varepsilon \geq 0 \tag{13}$$

and the manifold:

$$M_0 = \{y = z = 0, x \in [-1, 1]\} \tag{14}$$

## 2.1 Normally hyperbolic invariant manifold

For $\varepsilon = 0$, while denoting the right hand side of eq. (13) with $\mathbf{f}(x, y, z)$, we can see that

$$\mathbf{f}(x, y, z)|_{M_0} = \mathbf{f}(x, 0, 0)|_{x \in [-1,1]} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{15}$$

$M_0$ is therefore a normally hyperbolic invariant manifold, since the zero vector is tangent to any other vector. Note that $M_0$ is neither inflowing nor overflowing.

## 2.2 Perturbed manifold

The manifold stays the same, even after perturbation:

$$M_\varepsilon = \{y = z = 0, x \in [-1, 1]\} \tag{16}$$

We can verify this, by evaluating the right hand side $\mathbf{f}(x, y, z)$ for $\varepsilon > 0$ on $M_\varepsilon$:

$$\mathbf{f}(x, y, z)|_{M_\varepsilon} = \mathbf{f}(x, 0, 0)|_{x \in [-1,1]} = \begin{pmatrix} -\varepsilon x \\ 0 \\ 0 \end{pmatrix} \parallel \mathbf{v} \qquad \forall \, \mathbf{v} \in M_\varepsilon$$

In other words, there will be no component sticking out from our manifold, even when $\varepsilon > 0$. It is interesting to note, that the perturbed manifold is now *inflowing* invariant.


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## 2.3 Stable fibers

To apply the property (v), we need first to get an analytic expression of the flow map:

$$\begin{aligned}
\dot{z} = z \quad &\Rightarrow \quad z(t) = z_0\, e^{t} \\
\dot{y} = y \quad &\Rightarrow \quad y(t) = y_0\, e^{-t} \\
\Rightarrow \quad \dot{x} &= -\varepsilon \left(x + y_0^2\, e^{-2\,t}\right) \\
\dot{x} + \varepsilon\, x &= -\varepsilon\, y_0^2\, e^{-2\,t} \\
x_{hom}(t) &= c_0\, e^{-\varepsilon\, t} \\
x_p(t) &= c_1\, e^{-2\,t} \\
\Rightarrow -2\,t\,c_1\, e^{-2\,t} + \varepsilon\, c_1\, e^{-2\,t} = -\varepsilon\, y_0^2\, e^{-2t} \quad &\Rightarrow \quad c_1 = \frac{y_0^2\, \varepsilon}{2-\varepsilon} \\
\Rightarrow x(0) = x_{hom}(0) + x_p(0) = c_0 + \frac{y_0^2\, \varepsilon}{2-\varepsilon} \quad &\Rightarrow \quad c_0 = x_0 - \frac{y_0^2\, \varepsilon}{2-\varepsilon}
\end{aligned}$$

Putting things together yields:

$$F^t(x_0, y_0, z_0) = \begin{bmatrix} \left(x_0 - \frac{y_0^2\,\varepsilon}{2-\varepsilon}\right) e^{-\varepsilon\, t} + \frac{y_0^2\,\varepsilon}{2-\varepsilon} e^{-2\,t} \\ y_0\, e^{-t} \\ z_0\, e^{t} \end{bmatrix} \tag{17}$$

From the flow map especially, it becomes quite clear that the $y$-direction will be contained in the stable subspace, and the $z$-direction in the unstable subspace, being both tangential to their local stable and unstable manifolds.

For the stable fibers, we can therefore clearly put $z = 0$ and evaluate the expression:

$$\begin{aligned}
&\mathbf{p} = (x_p, 0, 0) \in M \\
&\mathbf{q} = (x, y, 0) \\
&\hat{\mathbf{q}} = (\hat{x}, \hat{y}, 0) \\
\Rightarrow \quad &\lim_{t\to\infty} \frac{\left\| \begin{bmatrix} \left[\left(x - \frac{y^2\,\varepsilon}{2-\varepsilon}\right) - x_p\right] e^{-\varepsilon\, t} + \frac{y}{2-\varepsilon}\, e^{-2\,t} \\ y\, e^{-t} \end{bmatrix} \right\|}{\left\| \begin{bmatrix} \left[\left(\hat{x} - \frac{\hat{y}^2\,\varepsilon}{2-\varepsilon}\right) - x_p\right] e^{-\varepsilon\, t} + \frac{\hat{y}}{2-\varepsilon}\, e^{-2\,t} \\ \hat{y}\, e^{-t} \end{bmatrix} \right\|} \stackrel{?}{=} 0
\end{aligned}$$

All the involved terms go to zero, for $t \to \infty$, but at different pace. $e^{-\varepsilon\, t}$ shows the slowest convergence, whereas $e^{-2\,t}$ the fastest. So if we want to guarantee that the limit converges to zero, we need to make sure that the numerator does so much faster than the denominator. If we therefore impose the vanishing of the term $e^{-\varepsilon\, t}$ in the numerator, while in the denominator it persists, through the hypothesis $\mathbf{q} \neq \hat{\mathbf{q}}$, we get:

$$\left(x - \frac{y^2\,\varepsilon}{2-\varepsilon}\right) - x_p = 0 \Rightarrow \quad x = x_p + \frac{y^2\,\varepsilon}{2-\varepsilon}$$


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Yielding the expression for the stable fiber:

$$f^s(\mathbf{p}) = \left\{(x, y, z) \mid x = x_p + \frac{y^2\,\varepsilon}{2-\varepsilon}, z = 0\right\} \tag{18}$$

□ *q.e.d.*

## 2.4 Unstable fibers

For the unstable fibers, as mentioned before already, we can put $y = 0$, and have to consider $z \neq 0$. This leads to:

$$\begin{aligned} \mathbf{p} &= (x_p, 0, 0) \in M \\ \mathbf{q} &= (x, 0, z) \\ \hat{\mathbf{q}} &= (\hat{x}, 0, \hat{z}) \\ \Rightarrow \quad \lim_{t\to-\infty} & \frac{\left\| \begin{bmatrix} (x - x_p)\, e^{-\varepsilon t} \\ z\, e^t \end{bmatrix} \right\|}{\left\| \begin{bmatrix} (\hat{x} - x_p)\, e^{-\varepsilon t} \\ \hat{z}\, e^t \end{bmatrix} \right\|} \overset{?}{=} 0 \end{aligned}$$

The only chance for this to converge to zero is granted by:

$$x = x_p$$

as otherwise the term $e^{-\varepsilon t}$ diverges in backward time. The unstable fiber expression is therefore:

$$f^u(\mathbf{p}) = \left\{(x, y, z) \mid x = x_p, y = 0\right\} \tag{19}$$

## 2.5 Properties verification

**(i) union to manifold** Recall the stable fiber with base point $\mathbf{p}$:

$$f^s(\mathbf{p}) = \left\{(x, y, z) \mid x = x_p + \frac{y^2\,\varepsilon}{2-\varepsilon}, z = 0\right\}$$

Inspection of the structure makes clear that for a fixed $x_p \in M$, the equation draws a quadratic curve in $x$-$y$ - plane, with the leading coefficient $\varepsilon$. This means that the straight fibers for $\varepsilon = 0$, turn into curved ones for small $\varepsilon$. This is exactly how the $W^s_{loc}(M)$ perturbs, without leaving the stable subspace $x$, $y$ for small $\varepsilon$, it yields a flat surface with curvy edges:

$$W^s_{loc}(M) = \bigcup_{p\in M} f^s(\mathbf{p}) = W^s_{loc} = \left\{(x, y, z) \mid -1 + \frac{y^2\,\varepsilon}{2-\varepsilon} \leq x \leq 1 + \frac{y^2\,\varepsilon}{2-\varepsilon}, z = 0\right\} \subset E^s$$

Note that even if the "true" $W^s(M)$ did not perturbed in this way, and actually remained a straight surface, we would always be able to find a *finite* surface on which the approximation holds:


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$$\exists\, \delta > 0 : \quad (-1,1) \times [-\delta, \delta] \times \{0\} \subset \bigcup_{p \in M} f^s(\mathbf{p})$$

**(ii) tangency** Considering $N_p^s$, the normal projection from our manifold $M$ would generally go into our $y$ and $z$ direction, however the stable subspace is only along the $y$ direction, since $z(t) \neq= 0$ diverges. The sub-bundle $N^s M$ in our case is therefore the simply given by the $x$-$y$-plane. Since $f^s(\mathbf{p})$ is contained in that plane it is also tangent.

**(iii) positively invariant family** we need to show that

$$F^t(f^s(\mathbf{p})) \subset f^s(F^t((\mathbf{p})) \tag{20}$$

we recall that for $\mathbf{p}$ to be a base point of the stable fiber, it is necessary that $\mathbf{p} \in M$, therefore we can focus on the $x_p = x_0$ coordinate. For the left hand side of the statement we have:

$$\begin{aligned} x_1 &= x_0 + \frac{y_0^2\,\varepsilon}{2-\varepsilon} \\ x_2 &= \left(x_1 - \frac{y_0^2\,\varepsilon}{2-\varepsilon}\right) e^{-\varepsilon\, t} + \frac{y_0^2\,\varepsilon}{2-\varepsilon} e^{-2\,t} \\ &= x_0\, e^{-\varepsilon\, t} + \frac{y_0^2\,\varepsilon}{2-\varepsilon} e^{-2\,t} \end{aligned}$$

for the right hand side we get:

$$\begin{aligned} x_1 &= \left(x_0 - \frac{y_0^2\,\varepsilon}{2-\varepsilon}\right) e^{-\varepsilon\, t} + \frac{y_0^2\,\varepsilon}{2-\varepsilon} e^{-2\,t} \\ x_2 &= x_1 + \frac{y_1^2\,\varepsilon}{2-\varepsilon} \\ &= \left(x_0 - \frac{y_0^2\,\varepsilon}{2-\varepsilon}\right) e^{-\varepsilon\, t} + \frac{y_0^2\,\varepsilon}{2-\varepsilon} e^{-2\,t} + \frac{y_0^2\,\varepsilon}{2-\varepsilon} e^{-2t} \end{aligned}$$

Since with $p \in M$, we have $y_0 = 0$, we get:

$$x_0\, e^{-\varepsilon\, t} = x_0\, e^{-\varepsilon\, t} \tag{21}$$

□ *q.e.d.*

**(iv) exponential upper bound**

$$\|F^t(\mathbf{q}) - F^t(\mathbf{p})\| \overset{?}{<} C_s e^{-\lambda_s\, t} \qquad \mathbf{q} \in f^s(\mathbf{p}) \tag{22}$$

Since $\mathbf{p} \in M$, the flowmap produces just the reduced dynamics, and we can simplify further by restricting the flow map on the stable fiber for $\mathbf{q}$ (canceling out the $y_0$ fraction), this yields:


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$$\|F^t(\mathbf{q}) - F^t(\mathbf{p})\| =$$

$$\left\| \begin{bmatrix} x_p e^{-\varepsilon t} + \frac{y_q^2 \varepsilon}{2-\varepsilon} e^{-2t} \\ y_q e^{-t} \end{bmatrix} - \begin{bmatrix} x_p e^{-\varepsilon t} \\ 0 \end{bmatrix} \right\| =$$

$$\left\| \begin{bmatrix} \frac{y_q^2 \varepsilon}{2-\varepsilon} e^{-2t} \\ y_q e^{-t} \end{bmatrix} \right\| \leq$$

$$\left\| \frac{y_q^2 \varepsilon}{2-\varepsilon} e^{-2t} \right\| + \left\| y_q e^{-t} \right\| \leq \left( \frac{y_q^2 \varepsilon}{2-\varepsilon} + |y_q| \right) e^{-t}$$

□ *q.e.d.*

**(v)** see section 2.3.

**(vi) fibers do not intersect** We proof ad absurdum. Recalling

$$f^s(\mathbf{p}) = \left\{ (x, y, z) \mid x = x_p + \frac{y^2 \varepsilon}{2-\varepsilon}, z = 0 \right\}$$

suppose now that two stable fibers with different base points $x_1 \neq x_2$ would at $y$ intersect:[1]

$$\Rightarrow \quad x_1 + \frac{y^2 \varepsilon}{2-\varepsilon} = x_2 + \frac{y^2 \varepsilon}{2-\varepsilon} \Rightarrow x_1 = x_2$$

which conflicts with the previous assumption.

□ *q.e.d.*

**(vii) Smooth function of basepoint** In our example the fibers depend linearly on their base-points:

$$x = x_p + \frac{y^2 \varepsilon}{2-\varepsilon} \tag{23}$$

they are indeed $C^\infty$ in $x_p$, as we would assume given that our system as well as our manifold are $C^\infty$ in $x$.

## 2.6 Local manifolds

Trajectories on the manifold are given by the reduced dynamics:

$$x(t) = x_0 e^{-\varepsilon t} \tag{24}$$

Note that depending on the sign of $x_0$, we might get a trajectory with positive or negative $x(t)$. Together with the fixed point $x_0 = 0$, there are hence 3 distinct trajectories on the manifold:

$$\gamma(\mathbf{p}) = \{F^t(\mathbf{p})\}_{t \geq 0} = \begin{cases} [x_p, 0) & \text{for } x_p < 0 \\ 0 & \text{for } x_p = 0 \\ (0, x_p] & \text{for } x_p > 0 \end{cases} \qquad \mathbf{p} \in M \Rightarrow x_p \in [-1, 1] \tag{25}$$

[1] Please note that the y-coordinates are the same, since we are assuming intersection it follows $y_1 = y_2 = y$


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In this way we can construct the local manifolds as follows:

$$W^{ss}_{loc}(\gamma(\mathbf{p})) = \begin{cases} \left\{(x,y,z) \mid x = \frac{y^2\varepsilon}{2-\varepsilon}, z = 0\right\} & \text{for } x_p = 0 \\ \left\{(x,y,z) \mid 0 < x \le x_p + \frac{y^2\varepsilon}{2-\varepsilon}, z = 0\right\} & \text{for } x_p > 0 \\ \left\{(x,y,z) \mid 0 > x \ge x_p + \frac{y^2\varepsilon}{2-\varepsilon}, z = 0\right\} & \text{for } x_p < 0 \end{cases} \tag{26}$$

Note that in our example, the super stable local manifold, and the stable local manifold are the same, since there is no dimension left in the manifold, from which trajectories could be attracted, in other words $\dim \gamma(\mathbf{p}) = \dim M$. Considering the trajectory as a manifold and constructing the local extensions by fibers will therefore generate the same local stable manifold[2], in our example:

$$W^{s}_{loc}(\gamma(\mathbf{p})) = W^{ss}_{loc}(\gamma(\mathbf{p})) \tag{27}$$

We proceed in a similar way for the unstable manifolds, with some simplifications in the expressions, given the less involved construction of $f^u(\mathbf{p})$:

$$W^{uu}_{loc} = \begin{cases} \{(x,y,z) \mid x = 0, y = 0\} & \text{for } x_p = 0 \\ \left\{(x,y,z) \mid 0 < x \le x_p, y = 0\right\} & \text{for } x_p > 0 \\ \left\{(x,y,z) \mid 0 > x \ge x_p, y = 0\right\} & \text{for } x_p < 0 \end{cases} \tag{28}$$

and finally, as well for the unstable parts,

$$W^{u}_{loc}(\gamma(\mathbf{p})) = W^{uu}_{loc}(\gamma(\mathbf{p})) \tag{29}$$

[2]Note that in one case we treat the trajectory as a new manifold, where the projection into the normal space could be in any direction, whereas in the super stable case we are still considering the fibers sticking out from the same manifold, just with basepoints restricted to our trajectory


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