Nonlinear Dynamics and Chaos II. Assignment 2

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1 Coupled pendulums

By using the angles α and β as defined in the problem statement, and adding a coordinate system with y-axis pointing *downwards*, and x-axis pointing to the left, we can write out the coordinates of the two masses m as:

$$\mathbf{r}_1 = L \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} \tag{1}$$

$$\mathbf{r}_2 = L \begin{pmatrix} \sin\alpha + \sin\beta \\ \cos\alpha + \cos\beta \end{pmatrix} \tag{2}$$

By considering the y coordinate to write out the potential energy of the two equally dimensioned masses m, we get:

$$V = mgL(3 - 2\cos\alpha - \cos\beta) \tag{3}$$

Note that the constant term $3mgL$ would just set the energy level to zero in the neutral position, but has no mathematical impact, since potentials may be shifted by a constant.

From the simple pendulum we know already, that $T_1 = \frac{1}{2}mL^2\dot{\alpha}^2$, in order to calculate T_2, we first evaluate

$$\begin{aligned} |\dot{\mathbf{r}}_2|^2 &= \left(L(\dot{\alpha}\cos\alpha + \dot{\beta}\cos\beta)\right)^2 + \left(L(-\dot{\alpha}\sin\alpha - \dot{\beta}\sin\beta)\right)^2 \\ &= L^2\left(\dot{\alpha}^2 + 2\dot{\alpha}\dot{\beta}\cos(\alpha-\beta) + \dot{\beta}^2\right) \end{aligned}$$

Note that the cosine of the sum of two angles was used to simplify the mixed terms.

All together yields then the kinetic energy:

$$T = \frac{1}{2}mL^2\left(2\dot{\alpha}^2 + 2\dot{\alpha}\dot{\beta}\cos(\alpha-\beta) + \dot{\beta}^2\right) \tag{4}$$

The lagrangian is then simply given by

$$\mathcal{L} = T - V = \frac{1}{2}mL^2\left(2\dot{\alpha}^2 + 2\dot{\alpha}\dot{\beta}\cos(\alpha-\beta) + \dot{\beta}^2\right) - mgL(3 - 2\cos\alpha - \cos\beta) \tag{5}$$

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Now we can calculate the generalized momenta by $\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}$, where in our case $\dot{\mathbf{q}} = \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix}$. This means that only the kinetic energy is of importance in this operation. We get:

$$\begin{cases} p_1 = mL^2 \left(2\dot{\alpha} + \cos(\alpha - \beta)\dot{\beta}\right) \\ p_2 = mL^2 \left(\cos(\alpha - \beta)\dot{\alpha} + \dot{\beta}\right) \end{cases}$$

This can be perhaps more clearly written in matrix form:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = mL^2 \begin{bmatrix} 2 & \cos(\alpha - \beta) \\ \cos(\alpha - \beta) & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} \tag{6}$$

We define the matrix M:

$$M \coloneqq mL^2 \begin{bmatrix} 2 & \cos(\alpha - \beta) \\ \cos(\alpha - \beta) & 1 \end{bmatrix} \tag{7}$$

Reconsidering eq. (4) it is easy to see that the kinetic energy is substantially a quadratic form governed by M:

$$T = \frac{1}{2} \begin{bmatrix} \dot{\alpha} & \dot{\beta} \end{bmatrix} M \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} \tag{8}$$

For being able to write out the Hamiltonian all that is left to do, is writing this quadratic form in terms of the general momenta. Using $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = M \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}$ from eq. (6), it follows mathematically:

$$\begin{aligned} T &= \frac{1}{2} \begin{bmatrix} \dot{\alpha} & \dot{\beta} \end{bmatrix} M \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \dot{\alpha} & \dot{\beta} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\ &= \frac{1}{2} \left(M^{-1} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \right)^\top \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} p_1 & p_2 \end{bmatrix} M^{-1} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \end{aligned}$$

Using the fact that

$$\left(M^{-1}\right)^\top = M^{-1} \quad \iff \quad M^\top = M$$

By working out the details, we get:

$$M^{-1} = \frac{1}{mL^2 \left(2 - \cos^2(\alpha - \beta)\right)} \begin{bmatrix} 1 & -\cos(\alpha - \beta) \\ -\cos(\alpha - \beta) & 2 \end{bmatrix}$$

We then get for the Hamlitonian:

$$\mathcal{H} = T + V = \frac{1}{2} \frac{p_1^2 - 2p_1 p_2 \cos(\alpha - \beta) + 2p_2^2}{mL^2 \left(2 - \cos^2(\alpha - \beta)\right)} + mgL(3 - 2\cos\alpha - \cos\beta)$$

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More precisely we can substitute (α, β) by (q_1, q_2) to get the proper Hamiltonian:

$$\mathcal{H}(q_1, q_2, p_1, p_2) = \frac{p_1^2 - 2p_1p_2\cos(q_1 - q_2) + 2p_2^2}{2mL^2\,(2 - \cos^2(q_1 - q_2))} + mgL(3 - 2\cos q_1 - \cos q_2) \tag{9}$$

The system equations are then obtained by:

$$\dot{\mathbf{q}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}$$
$$\dot{\mathbf{p}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}}$$

This yields:

$$\begin{cases} \dot{q}_1 &= \dfrac{2p_1 - 2p_2\cos(q_1 - q_2)}{2L^2m\,(2 - \cos^2(q_1 - q_2))} \\ \dot{q}_2 &= \dfrac{4p_2 - 2p_1\cos(q_1 - q_2)}{2L^2m\,(2 - \cos^2(q_1 - q_2))} \\ \dot{p}_1 &= -\dfrac{\sin(q_1 - q_2)\,(p_1p_2\cos^2(q_1 - q_2) - (p_1^2 + 2p_2^2)\cos(q_1 - q_2) + 2p_1p_2)}{L^2m\,(2 - \cos^2(q_1 - q_2))^2} - 2gLm\sin(q_1) \\ \dot{p}_2 &= -\dfrac{\sin(q_1 - q_2)\,(-p_1p_2\cos^2(q_1 - q_2) + (p_1^2 + 2p_2^2)\cos(q_1 - q_2) - 2p_1p_2)}{L^2m\,(2 - \cos^2(q_1 - q_2))^2} - gLm\sin(q_2) \end{cases}$$

2 Lotka–Volterra model

Given the system

$$\begin{cases} \dot{h} = a_1h(1 - bp) \\ \dot{p} = -a_2p(1 - ch) \end{cases} \qquad a_1, a_2, b, c > 0 \tag{10}$$

By assuming also $p > 0$ and $h > 0$, we can rewrite the system as follows:

$$\begin{cases} \dot{h} = h\,p\,a_1\left(\dfrac{1}{p} - b\right) \\ \dot{p} = -h\,p\,a_2\left(\dfrac{1}{h} - c\right) \end{cases} \tag{11}$$

The common factor $h\,p$ can be canceled out by using the appropriate rescaling of time:

$$\begin{aligned} \tau &= \int_0^t h(s)\,p(s)ds \\ \frac{d\tau}{dt} &= h\,p \\ \frac{d(\bullet)}{dt} &= \frac{d(\bullet)}{d\tau}\frac{d\tau}{dt} = h\,p\,\frac{d(\bullet)}{d\tau} \end{aligned} \tag{12}$$

For ease of notation we introduce

$$(\bullet)' \coloneqq \frac{d(\bullet)}{d\tau} \tag{13}$$

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$$\begin{cases} h\,p\,h' = h\,p\,a_1\left(\frac{1}{p} - b\right) \\ h\,p\,p' = -h\,p\,a_2\left(\frac{1}{h} - c\right) \end{cases}$$

As anticipated the common terms can be simplified, always assuming $h, p > 0$, yielding then:

$$\begin{cases} h' = a_1\left(\frac{1}{p} - b\right) \\ p' = -a_2\left(\frac{1}{h} - c\right) \end{cases}$$

Given that the right hand side depends just on p and q respectively, it is quite straight forward to construct a Hamiltonian for this system:

$$\mathcal{H}(h, p) = a_1\,(\log(p) - b\,p) + a_2\,(\log(h) - h\,c) \tag{14}$$

where log denotes the natural logarithm. The canonical Hamiltonian system is then obtained by:

$$h' = \frac{\partial \mathcal{H}}{\partial p} = a_1\left(\frac{1}{p} - b\right)$$
$$p' = -\frac{\partial \mathcal{H}}{\partial h} = -a_2\left(\frac{1}{h} - c\right)$$

The fixed point for this system, for the constraint that $h, p > 0$ is given by:

$$(h^\star, p^\star) = \left(\frac{1}{c}, \frac{1}{b}\right) \tag{15}$$

To establish stability we need to evaluate the definiteness of the Hessian of $\mathcal{H}$ in this fixed point:

$$\mathrm{D}^2\mathcal{H}(h, p)\Big|_{(h^\star, p^\star)} = \begin{bmatrix} -\frac{a_2}{h^2} & 0 \\ 0 & -\frac{a_1}{p^2} \end{bmatrix}\Bigg|_{(h^\star, p^\star)} = \begin{bmatrix} -a_2\,c^2 & 0 \\ 0 & -a_1\,b^2 \end{bmatrix} \prec 0 \tag{16}$$

We propose as Lyapunov function:

$$V(h, p) = -\mathcal{H}(h, p) = a_1\,(b\,p - \log(p)) + a_2\,(h\,c - \log(h)) \tag{17}$$

which shows the following properties:

- $\mathrm{D}V(h^\star, p^\star) = 0$
- $\frac{dV(h(t), q(t))}{dt} = 0$
- $\mathrm{D}^2V(h^\star, p^\star) \succ 0$

This allows us to conclude *nonlinear stability* by Lyapunov's direct method.

□ *q.e.d.*

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3 Steady compressible fluid flows

The general equation of continuity, which guarantees that masses cannot simply disappear, but have to evolve in a continuous fashion, reads:

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{18}$$

Since steady flow implies, that the fluid flow is in a steady state, where it does not change with time anymore, we can simplify the continuity equation by noting that also the density ρ must not change with time anymore. However, it well may change in space, since we assume *compressibility*. Equation (18) becomes:

$$\begin{aligned} \nabla \cdot (\rho \mathbf{v}) = 0 \\ \frac{\partial}{\partial x}(\rho\, u(x,y)) + \frac{\partial}{\partial y}(\rho\, v(x,y)) = 0 \end{aligned} \tag{19}$$

As we have seen in class this can be expressed by using the three dimensional cross product as well:

$$\nabla \times \left(\rho \begin{bmatrix} -v(x,y) \\ u(x,y) \\ 0 \end{bmatrix} \right) = \mathbf{0} \tag{20}$$

Note that the resulting vector is zero in the the directions x and y, since $u(x,y)$ and $v(x,y)$ do not depend on z. The z component reassembles eq. (19).

The theorem of Stokes, applied to the above vector field, guarantees that any closed line integral in space is zero, i.e. it is conservative. Conservative fields admit a potential, denoting it by Ψ yields:

$$\begin{cases} -\rho\, v(x,y) = \dfrac{\partial \Psi(x,y)}{\partial x} \\ \rho\, u(x,y) = \dfrac{\partial \Psi(x,y)}{\partial y} \end{cases}$$

Rearranging by assuming $\rho(x,y) > 0$, yields:

$$\begin{cases} u(x,y) = \dfrac{1}{\rho}\dfrac{\partial \Psi(x,y)}{\partial y} \\ v(x,y) = -\dfrac{1}{\rho}\dfrac{\partial \Psi(x,y)}{\partial x} \end{cases} \tag{21}$$

The general equation of motion for floating particles is governed by this velocity field:

$$\begin{cases} \dot{x} = u(x,y) \\ \dot{y} = v(x,y) \end{cases} \tag{22}$$

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Similar to the derivation provided in eq. (12), we introduce the new time τ, and its derivative by $(\bullet)'$, such that:

$$\begin{aligned} \tau &= \int_0^t \frac{1}{\rho(x(s), y(s))} ds \\ \frac{d(\bullet)}{dt} &= (\bullet)' \frac{d\tau}{dt} = \frac{1}{\rho}(\bullet)' \end{aligned} \tag{23}$$

Rescaling the equation of motion in this way, while substituting the stream function Ψ yields:

$$\begin{cases} \frac{1}{\rho} x' = \frac{1}{\rho} \frac{\partial \Psi(x, y)}{\partial y} \\ \frac{1}{\rho} y' = -\frac{1}{\rho} \frac{\partial \Psi(x, y)}{\partial x} \end{cases} \tag{24}$$

Which then becomes the canonical Hamiltonian system, we have been seeking for:

$$\begin{cases} x' = \frac{\partial \Psi(x, y)}{\partial y} \\ y' = -\frac{\partial \Psi(x, y)}{\partial x} \end{cases} \tag{25}$$

with the Hamiltonian $\mathcal{H}(x, y) = \Psi(x, y)$.

□ *q.e.d.*

4 Two dimensional torus phase space

Consider the following system:

$$\begin{cases} \ddot{x} + x = 0 \\ \ddot{y} + y = 0 \end{cases} \tag{26}$$

By the general solution theory for ordinary differential equations, it easy to show that the most general solution is given by:

$$\begin{cases} x(t) = c_1 \cos(t) + c_2 \sin(t) \\ y(t) = c_3 \cos(t) + c_4 \sin(t) \end{cases} \tag{27}$$

With the usage of trigonometric identities we can rewrite those general solutions as phase shifted sine functions:

$$\begin{aligned} x(t) &= a_1 \sin(t + \theta_1) \\ &= a_1 \sin \theta_1 \cos t + a_1 \cos \theta_1 \sin t \\ &\Rightarrow \quad \begin{cases} a_1 \sin \theta_1 = c_1 \\ a_1 \cos \theta_1 = c_2 \end{cases} \quad \Rightarrow \quad \begin{cases} \theta_1 = \operatorname{arctan2}(c_1, c_2) \\ a_1 = \sqrt{c_1^2 + c_2^2} \end{cases} \end{aligned}$$

The most general solutions therefore become:

$$\begin{cases} x(t) = a_1 \sin(t + \theta_1) \\ y(t) = a_2 \sin(t + \theta_2) \end{cases} \tag{28}$$

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Note that the solutions are coming from a second order ordinary differential equation, which in general would be decomposed into a system of two coupled first order ODE, such that:

$$\left.\begin{aligned} \ddot{x} + x = 0 \quad &\Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \\ \ddot{y} + y = 0 \quad &\Rightarrow \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \\ \dot{y}_1 = y_2 \\ \dot{y}_2 = -y_1 \end{cases} \tag{29}$$

We can therefore simply derive the general solutions given in eq. (28) to obtain the complete dynamical system's solution:

$$\begin{cases} x_1(t) = a_1 \sin(t + \theta_1) \\ x_2(t) = a_1 \cos(t + \theta_1) \\ y_1(t) = a_2 \sin(t + \theta_2) \\ y_2(t) = a_2 \cos(t + \theta_2) \end{cases} \tag{30}$$

We can immediately see that the solutions are pairwise confined to a circle with center in the origin and radius a_1 and a_2 respectively. We may use the following definition of n-sphere of radius r:

$$S^n(r) = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \; : \; \|\mathbf{x}\| = r \right\} \tag{31}$$

The phase space of the overall system is confined $S^1(a_1) \times S^1(a_2)$, where a_1 and a_2 are the parameters coming from the initial conditions. Most importantly they do not vary and are constant. In other terms this means, that trajectories will be confined to two-dimensional invariant tori.

$$(\mathbf{x}(t), \mathbf{y}(t)) \in S^1(a_1) \times S^1(a_2) \subset \mathbb{R}^4 \qquad \forall\, t \in \mathbb{R}$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Such a phase space can be fully parameterized by just two *angles*. In our specific case we have:

$$\begin{cases} \varphi_1(t) = t + \theta_1 \\ \varphi_2(t) = t + \theta_2 \end{cases} \tag{32}$$

This means that we have reduced the dimensionality to two coordinates:

$$(\varphi_1(t), \varphi_2(t)) \in S^1 \times S^1 = \mathbb{T}^2 \tag{33}$$

Note that our oscillatory systems' equations in these new coordinates are simply given by:

$$\begin{cases} \dot{\varphi}_1 = 1 \\ \dot{\varphi}_2 = 1 \end{cases} \tag{34}$$

This system does not have any fixed point, whereas the 4-dimensional system in eq. (29) would admit the trivial fixed point **0**. This trivial fixed point would however require our torus construction to have radii zero, which clearly is a degenerate case.

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4.1 General case

A generalized version of the system is given by:

$$\begin{cases}\dot{\varphi}_1 = a(\varphi_1, \varphi_2) \\ \dot{\varphi}_2 = b(\varphi_1, \varphi_2)\end{cases} \qquad (\varphi_1, \varphi_2) \in \mathbb{T}^2 \tag{35}$$

we have to proof that this system *cannot* be Hamiltonian, if it does not admit any fixed point.

Ad absurdum assume that $\exists\, \mathcal{H}(\varphi_1, \varphi_2) \in C^1$ such that:

$$\begin{cases}\dot{\varphi}_1 = \dfrac{\partial \mathcal{H}}{\partial \varphi_2} \\ \dot{\varphi}_2 = -\dfrac{\partial \mathcal{H}}{\partial \varphi_1}\end{cases} \qquad (\varphi_1, \varphi_2) \in \mathbb{T}^2 \tag{36}$$

Not having a fixed point implies that:

$$\nabla \mathcal{H}(\varphi_1, \varphi_2) \neq \mathbf{0} \qquad \forall\, (\varphi_1, \varphi_2) \in \mathbb{T}^2 \tag{37}$$

For Weierstrass' extreme value theorem, $\mathcal{H}$ as a continuous function, has to show maximum and minimum on the closed compact set $\mathbb{T}^2$. Furthermore, the Hamiltonian is already restricted to this set, by depending exclusively on φ_1 or φ_2 both $\in S^1$. Therefore Fermat's necessary condition for a critical point must hold, i.e.

$$\exists (\varphi_{1_0}, \varphi_{2_0}) \;:\; \nabla \mathcal{H}(\varphi_{1_0}, \varphi_{2_0}) = \mathbf{0} \tag{38}$$

being a contradiction to the previous statement, which concludes our proof.

$\square$ *q.e.d.*

Rescaling time It is important to note that rescaling time in the system does not change anything in the above result. Assume a general common factor $s(\varphi_1, \varphi_2)$, such that:

$$\begin{cases}\dot{\varphi}_1 = a(\varphi_1, \varphi_2) = s(\varphi_1, \varphi_2)\, \tilde{a}(\varphi_1, \varphi_2) \\ \dot{\varphi}_2 = b(\varphi_1, \varphi_2) = s(\varphi_1, \varphi_2)\, \tilde{b}(\varphi_1, \varphi_2)\end{cases} \tag{39}$$

We can use such a factor to rescale time, as already shown before, in eqs. (12) and (23):

$$\begin{gathered}\tau = \int_0^t s(\varphi_1(s), \varphi_2(s)) ds \\ \frac{d(\bullet)}{dt} = (\bullet)' \frac{d\tau}{dt} = s(\varphi_1, \varphi_2)\, (\bullet)'\end{gathered} \tag{40}$$

Using this new time derivative notation we get:

$$\begin{cases}s(\varphi_1, \varphi_2)\, \varphi_1' = s(\varphi_1, \varphi_2)\, \tilde{a}(\varphi_1, \varphi_2) \\ s(\varphi_1, \varphi_2)\, \varphi_2' = s(\varphi_1, \varphi_2)\, \tilde{b}(\varphi_1, \varphi_2)\end{cases} \tag{41}$$

In order to simplify in the next step we need to guarantee that $s(\varphi_1, \varphi_2) \neq 0$, which then propagates the condition of not having a fixed point to $\tilde{a}$ and $\tilde{b}$ respectively:

$$s(\varphi_1, \varphi_2) \neq 0 \quad \Rightarrow \quad s(\varphi_1, \varphi_2)\, \tilde{a}(\varphi_1, \varphi_2) \neq 0 \iff \tilde{a}(\varphi_1, \varphi_2) \neq 0 \tag{42}$$

Therefore, the assumptions remain intact and the previous proof applies.

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5 Canonically conjugate variable

Given the dynamical system:

$$\dot{\mathbf{q}} = \mathbf{f}\left(\mathbf{q}, t\right) \qquad \mathbf{q} \in \mathbb{R}^n \tag{43}$$

We can construct a Hamiltonian for an extended system

$$\mathcal{H}(\mathbf{q}, \mathbf{p}, t) = \mathbf{p} \cdot \mathbf{f}\left(\mathbf{q}, t\right) \tag{44}$$

such that:

$$\begin{cases} \dot{\mathbf{q}} &= \dfrac{\partial \mathcal{H}}{\partial \mathbf{p}} = \mathbf{f}\left(\mathbf{q}, t\right) \\ \dot{\mathbf{p}} &= -\dfrac{\partial \mathcal{H}}{\partial \mathbf{q}} = -\dfrac{\partial \mathbf{f}\left(\mathbf{q}, t\right)}{\partial \mathbf{q}} \cdot \mathbf{p} \end{cases} \tag{45}$$

Note that the last product is between the Jacobian of **f** and the column vector **p**.

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