Nonlinear Dynamics and Chaos II. Assignment 1

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1 Chaos on binary extension map

Given the map

$$g(x) = 2x \bmod 1 \tag{1}$$

graphically shown in fig. 1

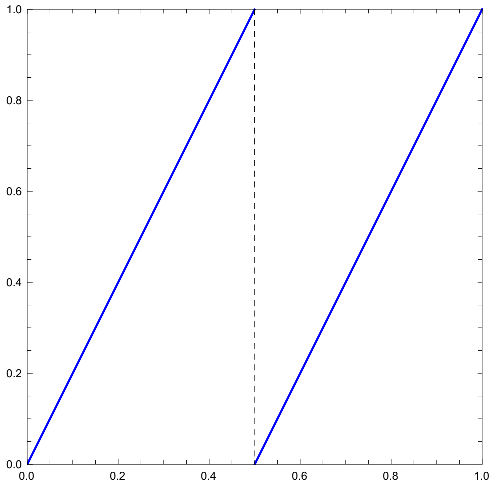

Figure 1: Binary extension map

we notice that this mapping can be used to convert any real number in the interval $[0,1)$ to a binary sequence, by choosing the digits b_i such that

$$x_0 = \sum_{i=1}^{\infty} \frac{b_i}{2^i}$$
$$x_i = g(x_{i-1}) \;\; \forall i > 0$$
$$b_i = \begin{cases} 0 & 2x_i < 1 \\ 1 & 2x_i \geq 1 \end{cases}$$

As we have shown in class already, there is an uncountable infinity of non-periodic sequences in this mapping, e.g. the trajectory starting with $x_0 = \frac{\pi}{6}$, which is one of the reasons we could

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show the Bernoulli shift map to be chaotic. Even though this binary mapping is unique and we could show that in this particular case applying the Bernoulli shift map actually equals applying the map given in eq. (1), we will rely again on showing topological conjugacy to the Bernoulli shift map, by introducing a general encoding for this map, relying on two closed intervals.

We formally define the semi-infinite sequence Σ an the mapping Ψ as:

$$\Sigma = \{.s_1 s_2 s_3 \ldots \mid s_i \in \{0,1\}\} \tag{2}$$

$$\Psi: \quad \Sigma \quad \to \quad [0,1] \tag{3}$$

$$.s_1 s_2 s_3 \ldots \mapsto x = \sum_{i=1}^{\infty} \frac{s_i}{2^i} \in I_{s_1} \cap g^{-1}(I_{s_2}) \cap g^{-2}(I_{s_3}) \cap g^{-3}(I_{s_4}) \cap \ldots$$

It is important to note that the whole domain $[0,1]$ of eq. (1) can be represented with this encoding, even the upper boundary $x = 1$: it is sufficient to choose all binary digits $s_i = 1$, i.e. $1.0 = 0.\bar{1} = \sum_{i=1}^{+\infty} \frac{1}{2^i}$.

Space partitioning The intervals I_{s_i} are either one of two *closed* intervals from the space partition:

$$[0,1] = [0,0.5] \cup [0.5,1.0] \tag{4}$$

The digit s_i indicates that $x_i \in [0,0.5]$ if $s_i = 0$, and $x_i \in [0.5,1]$ if $s_i = 1$.

Metric The metric in the symbolic space can be defined as:

$$d(s,\bar{s}) = \sum_{i=1}^{+\infty} \frac{|s_i - \bar{s}_i|}{2^i} \tag{5}$$

It is interesting to see that the measure of distance in this case serves as an upper bound for the absolute distance of the numbers x and $\bar{x}$ in $\mathbb{R}$ through the triangle inequality.[1]

Existence of infinite intersection Considering the map from eq. (1) we can show that:

$$I_{s_i} \cap g^{-1}(J) \neq \varnothing \qquad \forall\, J \subset [0,1] \text{ closed, non-empty, connected} \tag{6}$$

To imagine the pre-image $g^{-1}(J)$ it might be useful to consider fig. 1 at any single point $J = \{x_j\}$, go up to identity diagonal $x = 1$ and check on the horizontal direction the intersection with the graph $g(x)$, i.e. from where the trajectory could have come from. We easily can see that there are always two possibilities, one being from some point $\in [0,0.5]$ and the other coming from some point $\in [0.5,1]$. The intersection with I_{s_i} will therefore always be non-empty, closed and connected.

Applying $f(A \cap B) = f(A) \cap f(B)$ with A, B non-empty, closed and connected to our case, which is possible given that the chosen intervals are, together with the property from eq. (6), allows us to show in a recursive fashion that:

$$I_{s_1} \cap g^{-1}(I_{s_2}) \cap g^{-2}(I_{s_3}) \cap g^{-3}(I_{s_4}) \cap \ldots = I_{s_1} \bigcap_{i \geq 1} g^{-i}(I_{s_{i+1}}) \neq \varnothing$$

[1] The defined metric would be similar to an XOR operation on the binary sequence, which could be imagined as calculating the difference without taking care of the carry bit.

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Note that the notation $g^{-1}(I_{s_2})$ indicates thereby the pre-image of the mapping $g(x)$ (as opposed to the inverse of g, without issues on uniqueness). For instance, $g^{-1}([0.5,1]) = [0.25, 0.5] \cup [0.75, 1]$.

Equivalence of Bernoulli shift and map iteration By making again use of $f(A \cap B) = f(A) \cap f(B)$ with A, B non-empty, closed and connected, we can simply exploit that *by design*, a map iteration equals the Bernoulli shift:

$$\begin{aligned} x_1 = g(x_0) = g\left(I_{s_1} \bigcap_{i\geq 1} g^{-i}(I_{s_{i+1}}) \right) &= g(I_{s_1}) \bigcap_{i\geq 1} g^{-i+1}(I_{s_{i+1}}) = g(I_{s_1}) \cap I_{s_2} \bigcap_{i\geq 2} g^{-(i-1)}(I_{s_{i+1}}) \\ &= g(I_{s_1}) \cap I_{s_2} \bigcap_{i\geq 1} g^{-i}(I_{s_{i+2}}) = g(\tilde{I}_{s_0}) \cap \tilde{I}_{s_1} \bigcap_{i\geq 2} g^{-i}(\tilde{I}_{s_{i+1}}) = \tilde{I}_{s_1} \bigcap_{i\geq 1} g^{-i}(\tilde{I}_{s_{i+1}}) \end{aligned}$$

where $\tilde{I}$ denotes the intervals of the shifted Bernoulli number $\tilde{s}_i = s_{i+1}$. Note that the intersection with $g(\tilde{I}_{s_0})$ vanishes, because it spreads all over the interval $[0,1]$, which also explains why we can simply neglect the $\tilde{s}_0$ after a shift.

Uniqueness and invertibility Up to now, the encoding may seem rather unique. However, there is a countable infinity of trajectories that reach the fixed point 0, going through 0.5:

$$\begin{array}{l} x_1 \mapsto x_2 = g(x_1) \mapsto x_3 = (g \circ g)(x_1) \mapsto \dots \\ \left.\begin{array}{l} \left.\begin{array}{l} \frac{1}{8} \\ \frac{5}{8} \end{array}\right\} \mapsto \frac{1}{4} \\ \left.\begin{array}{l} \frac{3}{8} \\ \frac{7}{8} \end{array}\right\} \mapsto \frac{3}{4} \end{array}\right\} \mapsto \dfrac{1}{2} \mapsto 0 \end{array}$$

When considering to invert Ψ, the landing on exactly 0.5 creates an ambiguity which digit in the binary encoding to use, given that the space partition shares a common interval boundary at that point. By defining a new mapping $\tilde{\Psi}$, onto an interval $\tilde{I}$ excluding those trajectories, i.e.

$$\tilde{\Psi} : \Sigma \to \tilde{I} \qquad \tilde{I} = [0,1] \setminus \bigcup_{i\geq 0} g^{-i}\left(\left\{\frac{1}{2}\right\}\right) \tag{7}$$

we gain invertibility of $\tilde{\Psi}$. This actually means excluding all *finite* binary numbers (terminating with zeros), avoiding ambiguous representation in the infinite sequence, such as:

$$0.1 = 0.0\bar{1}$$

Conclusion Thanks to all $x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$, there is still an uncountable infinity of non-periodic orbits. By design of Ψ and $\tilde{\Psi}$ respectively, the Bernoulli-shift map on the semi-infinite binary sequence still serves well as a map iteration, as shown before. Therefore we can write:

$$\tilde{g} \circ \tilde{\Psi} = \tilde{\Psi} \circ \sigma \quad \Rightarrow \quad \tilde{g} = \tilde{\Psi} \circ \sigma \circ \tilde{\Psi}^{-1} \tag{8}$$

The binary extension map g is topologically semi-conjugate to the Bernoulli shift map σ on I, and on $\tilde{I}$ it is *topological conjugate*. $\tilde{I}$ simply excludes a *countable infinity* of rational numbers of the form $\frac{a}{2^n}$. It follows hereby that g is chaotic on $\tilde{I}$ and that it is therefore a chaotic map.

□ *q.e.d.*

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2 Bernoulli subshift properties

Given the bernoulli subshift of finite type:

$$\sigma : \Sigma_A^N \to \Sigma_A^N \tag{9}$$

Fixed point a fixed point is given by $s^\star = \sigma^k(s^\star) \quad \forall\, k \in \mathbb{N}$. In other words the symbol sequences always need to match regardless how often we shift to the left. This implies, that given two digits s_i and s_j we can select an arbitrary $k = j - i$, such that we must satisfy

$$s_i = s_j \quad \forall\ i, j \tag{10}$$

In the space Σ_A^N we have N different symbols, and therefore possible candidates for fixed points. However, the transition matrix A limits the allowed sequence from symbol number i to symbol number j by demanding $A_{ij} = 1$ (0 otherwise). For the N possible fixed points this means that additionally we must have $A_{ii} = 1$, which yields a number of $m \leq N$ fixed points:

$$m = \sum_{i=1}^{N} A_{ii} = \mathrm{Tr}(A) \tag{11}$$

$\square$ $q.e.d.$

k - periodic point k-periodic points (with a minimal period potentially $< k$) have to satisfy that:

$$\tilde{s} = \sigma^{nk}(\tilde{s}) \quad \forall\ n \in \mathbb{N} \tag{12}$$

This means that in the symbol space we must have a k-periodic sequence

$$\tilde{s} = \ldots s_i s_{i+1} s_{i+2} \ldots s_{i+k} \ldots \tag{13}$$

$$s_i = s_{i+k} \quad \forall\ i \in \mathbb{N} \tag{14}$$

In other words after k shifts of the digit s_j, we need to have $s_i = s_j$ again, with $k = j - i$. The transition matrix A limits the *admissible* orbits again, in particular it must be possible to get back to s_i again after k iterations, i.e. it is required that $(A^k)_{ii} \geq 1$.

Developing the power of the matrix unveils:

$$A^k = \underbrace{AAA\ldots}_{k-\text{times}} \qquad \text{for ease of notation, let } k = 4$$

$$(A^4)_{ij} = \sum_{l,m,n}^{N} a_{il} a_{lm} a_{mn} a_{nj}$$

$$\mathrm{Tr}(A^4) = \sum_i (A^4)_{ii} = \sum_{i,l,m,n}^{N} a_{il}\, a_{lm}\, a_{mn}\, a_{ni}$$

The term $a_{il}\, a_{lm}\, a_{mn}\, a_{ni}$ is indeed one, if and only if, the trajectory $< i, l, m, n, i >$ is admissible. So the trace is the sum of all possible periodic points. To illustrate this, let's assume the simplest case where all transitions are allowed, i.e. $A_{ij} = 1 \quad \forall i, j$, we have:

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$$\mathrm{Tr}(A^4) = \sum_{i,l,m,n}^{N} a_{il}\, a_{lm}\, a_{mn}\, a_{ni} = \sum_{i,l,m}^{N}\sum_{n}^{N} 1 = \sum_{i,l,m}^{N} N = \sum_{i,l}^{N} N^2 = \ldots = N^4$$
$$\Rightarrow \mathrm{Tr}(A^k) = N^k \qquad\qquad A_{ij} = 1 \quad \forall\, i, j$$

If we compare this with the result presented in the lectures of the number of period-k orbits:

$$N(k) = \frac{1}{k}\left(N^k - \sum_{<i,k>} N(k)\right) \qquad < i, k > i \text{ integer divider of } k \tag{15}$$

we notice that the $\mathrm{Tr}(A^k)$ in comparison shows:

(i) no correction for lower period orbits, given that the statement specifies that also periods $< k$ are accounted for

(ii) no division by k, because it treats, periodic *points* instead of *orbits*

Indeed a k periodic orbit contains k k-periodic points, which shows that for the special case $A_{ij} = 1 \quad \forall\, i, j$ indeed we get an equivalent result.
For the general case, we further develop the previous notation:

$$\begin{aligned}\mathrm{Tr}(A^k) = \sum_i (A^k)_{ii} &= \sum_i^N \left(\sum_{i_2, i_3, \ldots, i_k}^{N} A_{i i_2} A_{i_2 i_3} \ldots A_{i_{k-1} i_k A_{i_k i}} \right) \\ &= \sum_{i_1, i_2, \ldots, i_k} A_{i_1 i_2} \ldots A_{i_{k-1} i_k A_{i_k i_1}} \\ &= \sum_{i_1, \ldots, i_k} A_{i_k i_1} \prod_{l=2}^{k} A_{i_{l-1} i_l}\end{aligned}$$

It is natural to argue that the finite sequence $< i_l >\in \mathbb{N} \quad 1 \le l \le k$, for which $A_{i_k i_1} \prod_{l=2}^{k} A_{i_{l-1} i_l} = 1$, reassembles all admissible ways of going from i_1 to i_k with periodicity k and eventually even $< k$. As mentioned before, a single k-periodic *orbit* in this way will result into k k-periodic points. This results in the total number of admissible k-periodic points being:

$$\sum_{i=1}^{N} (A^k)_{ii} = \mathrm{Tr}(A^k) \tag{16}$$

□ $q.e.d.$

3 Lyapunov exponent for discrete dynamical systems

Given

$$\lambda(x_0) = \lim_{n \to +\infty} \frac{1}{n} \lim_{\delta \to 0^+} \log \frac{|f^n(x_0 + \delta) - f^n(x_0)|}{\delta} \tag{17}$$

Given that the innermost limit contains two continuous function, i.e. the logarithm and the absolute value, we may switch the order of the limit operation to the function evaluation. Note that in this way we extend the absolute value to the whole fraction, which is fine because the

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denominator δ is always positive, and hence $\delta = |\delta|$. This leads to the form given in eq. (18). Now the fraction inside the limit operation is eligible for the application of L'Hôpital's rule. By deriving with respect to δ, the chain rule lets a product sequence appear, see eq. (19). By exploiting the properties of the logarithm, we finally get eq. (20).

$$\begin{aligned}
\lambda(x_0) &= \lim_{n\to+\infty} \frac{1}{n} \lim_{\delta\to 0^+} \log \frac{|f^n(x_0+\delta) - f^n(x_0)|}{\delta} \\
&= \lim_{n\to+\infty} \frac{1}{n} \log \left| \lim_{\delta\to 0^+} \frac{f^n(x_0+\delta) - f^n(x_0)}{\delta} \right| && (18) \\
&= \lim_{n\to+\infty} \frac{1}{n} \log \left| \lim_{\delta\to 0^+} \frac{\prod_{i=0}^{n-1} f'\left(f^i\left(x_0+\delta\right)\right)}{1} \right| && (19) \\
&= \lim_{n\to+\infty} \frac{1}{n} \log \prod_{i=0}^{n-1} \left| f'\left(f^i\left(x_0\right)\right)\right| \\
&= \lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \left| f'\left(f^i\left(x_0\right)\right)\right| && (20)
\end{aligned}$$

□ *q.e.d.*

For the logistic map

$$f(x) = a\,x\,(1-x) \tag{21}$$

the dependence on initial conditions with $a \in [3,4]$ is given in fig. 2.

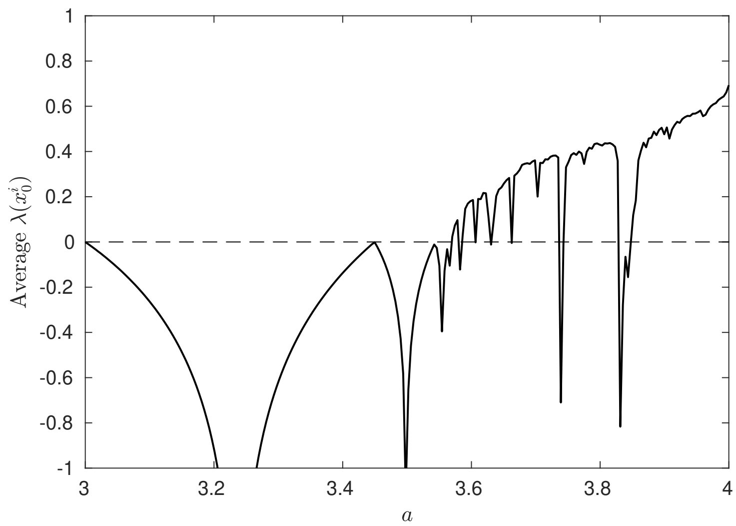

Figure 2: Average Lyapunov exponent against a

Any value a for which the Lyapunov exponent in the logarithmic version is > 0, two close initial conditions may diverge in exponential manner. From left to right, this is the case the first time slightly before 3.6. For $a = 4$ we have the highest Lyapunov exponent, indicating the possibility of chaos, which has indeed been shown to be the case.

Listing 1 shows the complete code used to generate fig. 2.

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```
%% parameter setup

a_lower = 3;
a_upper = 4;
num_values = 250;
num_iter = 10000;
num_initial_values = 10;

a = a_lower:(a_upper - a_lower)/(num_values-1):a_upper;
lambdas = zeros(1,length(a));

%% calculate lambda

for k = 1:length(a)
    sel_a = a(k);
    samples = zeros(1,num_initial_values);
    for i = 1:num_initial_values
        x = rand();
        for it = 1:num_iter
            samples(i) = samples(i) + log(abs(f_p(x,sel_a)));
            x = f(x,sel_a);
        end
        samples(i) = samples(i) / num_iter;
    end
    lambdas(k) = mean(samples);
end

%% plot

fig = figure;
ax = axes('Parent',fig);
plot(a,lambdas,'LineWidth', 1, 'Color', 'black');
hold on;
% dashed x-axis line
plot([a_lower a_upper], [0 0], 'Color', 'black', 'LineStyle', '--');
xlim([a_lower a_upper]);
ylim([-1 1]);
xlabel('$a$','Interpreter','latex');
ylabel('Average $\lambda(x_0^i)$','Interpreter','latex');

%% function definition

function x_next = f(x,a)
    x_next = a*x*(1-x);
end

function d = f_p(x,a)
    d = a - 2*a*x;
end
```

Listing 1: Lyapunov exponent calculation in Matlab