Nonlinear Dynamics and Chaos I. Assignment 6

Florian Mahlknecht

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1 Damped forced pendulum

Given the system

$$\ddot{x} + \delta\dot{x} + \sin x = \gamma \sin \omega t \qquad 0 < \delta \ll 1, \quad 0 < \gamma \ll 1 \tag{1}$$

we approximate for small oscillations:

$$\ddot{x} + \delta\dot{x} + x - \frac{1}{6}x^3 = \gamma \sin \omega t \tag{2}$$

Using a single harmonic for the harmonic balance method, we propose:

$$x_p(t) = c_1 \cos \omega t + s_1 \sin \omega t \tag{3}$$

Computing the third power we obtain:

$$\frac{1}{4}\left(3c_1^2 s_1 - s_1^3\right)\sin 3\omega t + \frac{3}{4}\left(c_1^2 s_1 + s_1^3\right)\sin \omega t + \frac{3}{4}\left(c_1 s_1^2 + c_1^3\right)\cos \omega t + \frac{1}{4}\left(c_1^3 - 3c_1 s_1^2\right)\cos 3\omega t \tag{4}$$

From this expression we neglect the 3rd harmonics, since we are interested in a single harmonic solution. In this way, eq. (2) with our chosen x_p becomes simply:

$$\cos \omega t\left(c_1 - c_1\omega^2 + s_1\delta\omega - \frac{1}{8}\left(c_1 s_1^2 + c_1^3\right)\right) + \sin \omega t\left(s_1 - s_1\omega^2 - c_1\delta\omega - \frac{1}{8}\left(c_1^2 s_1 + s_1^3\right)\right) = \gamma \sin \omega t \tag{5}$$

Now *balancing* this equation, i.e. comparing the coefficients of the the trigonometric functions, yields the following system of equations:

$$\begin{cases} c_1 - c_1\omega^2 + \delta s_1\omega - \frac{1}{8}\left(c_1 s_1^2 + c_1^3\right) = 0 \\ s_1 - s_1\omega^2 - \delta c_1\omega - \frac{1}{8}\left(c_1^2 s_1 + s_1^3\right) = \gamma \end{cases} \tag{6}$$

Changing to polar coordinates, i.e.

$$c_1 = A\cos\phi$$
$$s_1 = A\sin\phi$$

allow us to study the amplitude response of our proposed solution, given that eq. (3) becomes:

$$x_p(t) = A\cos\phi\cos\omega t + A\sin\phi\sin\omega t = A\cos(\omega t - \phi) \tag{7}$$

When substituting c_1 and s_1 in eq. (6), the arising higher order terms cancel out by using the trigonometric identity $cos^2\omega t + sin^2\omega t = 1$. We are left with:

$$\begin{cases} \left(A - A\omega^2 - \frac{1}{8}A^3\right)\cos\phi + A\delta\omega\sin\phi = 0 \\ \left(A - A\omega^2 - \frac{1}{8}A^3\right)\sin\phi - A\delta\omega\cos\phi = \gamma \end{cases} \tag{8}$$

By multiplying the first equation by $\cos\phi$, the second equation by $\sin\phi$ and then adding them up, we cancel out the second term:[1]

$$A - A\omega^2 - \frac{1}{8}A^3 = \gamma\sin\phi \tag{9}$$

By multiplying the first equation by $\sin\phi$, the second equation by $\cos\phi$ and then subtracting them, we cancel out the first term:

$$A\delta\omega = -\gamma\cos\phi \tag{10}$$

Squaring and adding eqs. (9) and (10) makes the final expression independent on ϕ:

$$\left(A - A\omega^2 - \frac{1}{8}A^3\right)^2 + A^2\delta^2\omega^2 - \gamma = 0 \tag{11}$$

which can be rewritten as a 6^{th} order polynomial in A:

$$\frac{1}{64}A^6 + \frac{1}{4}\left(\omega^2 - 1\right)A^4 + \left(\omega^4 + \delta^2\omega^2 - 2\omega^2 + 1\right)A^2 = \gamma^2 \tag{12}$$

For a fixed δ and γ and a range of values for ω, the roots can be numerically calculated. The real, positive roots, yielding the only valuable meaning for the amplitude A, are sketched in fig. 1, for $\delta = 0.1$.

[1] Note that this procedure implies that $\cos\phi \neq 0$ and $\sin\phi \neq 0$, i.e. we are restricting our domain to $\phi \neq k\frac{\pi}{2},\ k \in \mathbb{Z}$

 Florian Mahlknecht
fmahlknecht@student.ethz.ch
19-945-351

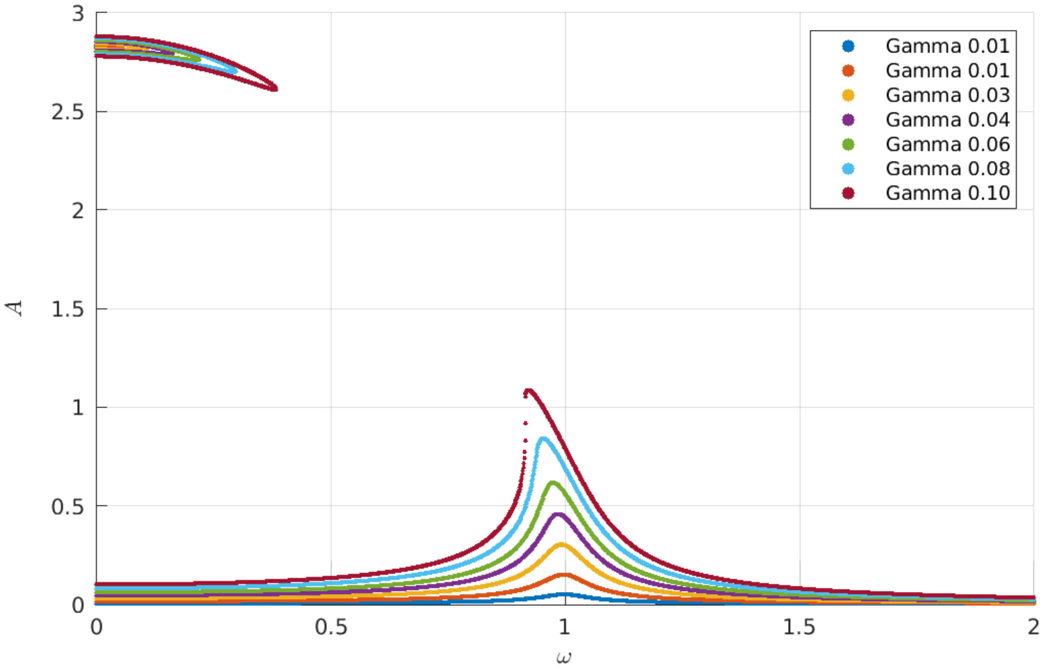

Figure 1: Amplitude behavior

Note that for large forcing, distinct solutions appear in the plot, which however are in lack of rigorous meaning, given that our assumption do not hold in those conditions anymore.

From the maxima, the backbone curve can be obtained, see fig. 2.

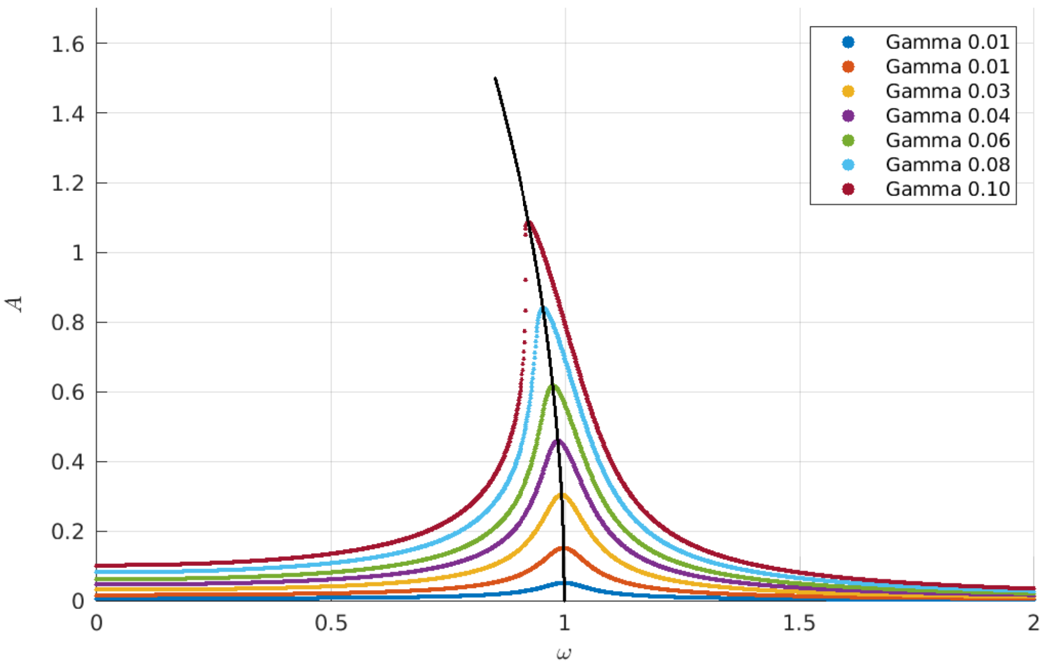

Figure 2: Backbone curve

fmahlknecht@student.ethz.ch
19-945-351

1.1 Heteroclinic orbits and their Melnikov functions

For $\delta = \gamma = 0$, we get the unperturbed, energy conserving system:

$$\begin{cases} \dot{\theta} = y \\ \dot{y} = -\sin\theta \end{cases} \tag{13}$$

The conserving quantity can be represented as potential and kinetic energy, e.g.:

$$H(\theta, y) = \frac{1}{2}y^2 + (1 - \cos\theta) \tag{14}$$

From the unstable fixed point $\theta = \pm\pi$, there is a pair of heteroclinic solutions departing and connecting, i.e. the separatrices. On those special trajectories $H(\theta, y) = 2$. In this way y can be isolated and plugged in the first equation of eq. (13), yielding:

$$\dot{\theta} = \pm\sqrt{2}\sqrt{1 + \cos\theta} \tag{15}$$

which is a separable ODE and can be shown to yield the following pair of heteroclinic orbits:

$$\begin{cases} \theta_h(t - t_0) = \pm 2\arctan\left(\sinh\left(t - t_0\right)\right) \\ y_h(t - t_0) = \pm 2\,\mathrm{sech}\left(t - t_0\right) \end{cases} \tag{16}$$

Before applying calculating the Melnikov function, it is important to remind that the perturbation in the original system given in eq. (1), for small values of δ and γ could always be rewritten as an ε perturbation:[2]

$$\begin{cases} \dot{\theta} = y \\ \dot{y} = -\sin\theta + \varepsilon\left(\gamma\sin\omega t - \delta y\right) \end{cases} \tag{17}$$

Given the definition of the Melnikov function:

$$M(t_0) = \int_{-\infty}^{+\infty} \left\langle \mathbf{f}^{\perp}\left(\mathbf{x}^0\left(t - t_0\right)\right), \mathbf{g}\left(\mathbf{x}^0\left(t - t_0\right), t\right)\right\rangle dt \tag{18}$$

we have:

$$\mathbf{f}^{\perp}\left(\mathbf{x}^0\left(t - t_0\right)\right) = \begin{pmatrix} \sin\left(\pm 2\arctan\left(\sinh\left(t - t_0\right)\right)\right) \\ \pm 2\,\mathrm{sech}\left(t - t_0\right) \end{pmatrix} \tag{19}$$

and

$$\mathbf{g}\left(\mathbf{x}^0\left(t - t_0\right), t\right) = \begin{pmatrix} 0 \\ \gamma\sin\omega t - \delta\left(\pm 2\,\mathrm{sech}\left(t - t_0\right)\right) \end{pmatrix} \tag{20}$$

which yields:

$$M(t_0) = \int_{-\infty}^{+\infty} \left[\pm 2\,\gamma\,\mathrm{sech}\left(t - t_0\right)\sin\omega t - 4\,\delta\,\mathrm{sech}^2\left(t - t_0\right)\right] dt \tag{21}$$

[2] For ease of notation we do not rescale them in a syntactically correct way

Florian Mahlknecht
fmahlknecht@student.ethz.ch
19-945-351

The second part of the integral can be simplified, given that $\int_{-\infty}^{+\infty} \text{sech}^2(t)dt = 2$. Therefore we have a transfer zero if and only if:

$$\frac{\delta}{\gamma} = \frac{1}{4}\int_{-\infty}^{+\infty} \pm\,\text{sech}\,(t - t_0)\sin \omega t\,dt \tag{22}$$

Numerical evaluation for $\omega = 2$ shows that the right hand side of eq. (22) is an oscillatory function bounded between ± 0.0677. See fig. 3.

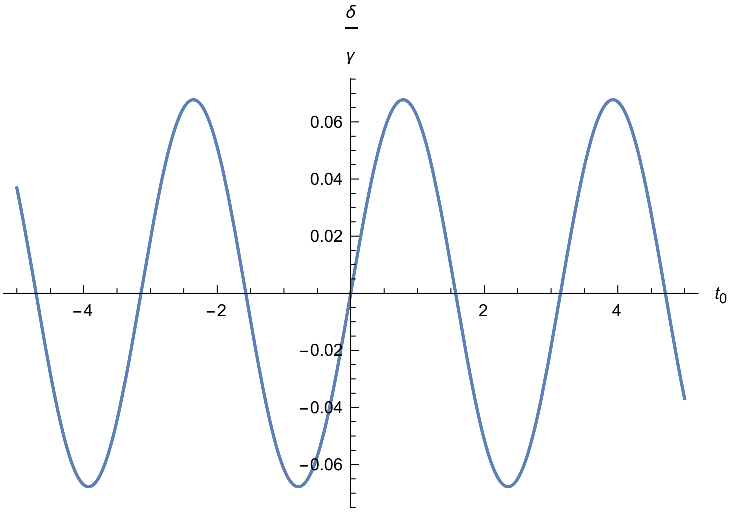

Figure 3: Numerical integration results

Thus, in order to observe chaotic behavior for any t_0, we need to choose the parameter ratio such that it is intersecting the curve, i.e. we need to guarantee:

$$\frac{\delta}{\gamma} < 0.0677 \tag{23}$$

This means that the damping term needs to be less than 6.7 % of the forcing term. Intuitively this makes sense, since we would expect sufficient damping to "normalize" the system's behavior.

Of course this can be translated in the desired $\frac{\gamma}{\delta}$ ratio, which looks as shown in fig. 4.

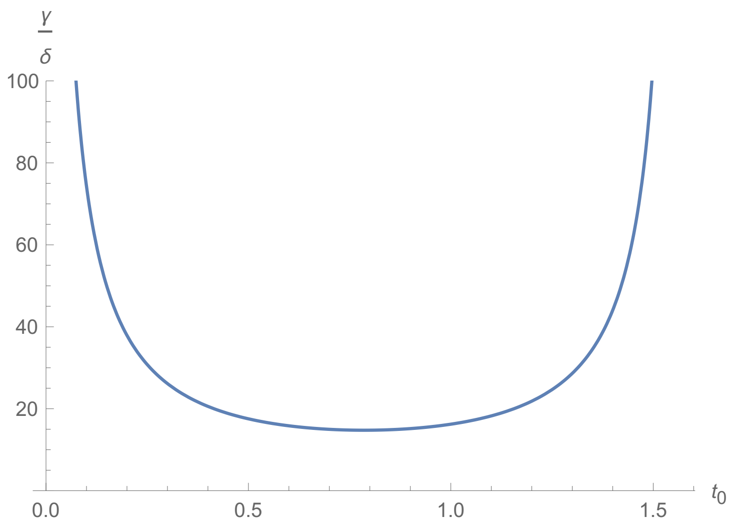

Figure 4: Parameter ratio

fmahlknecht@student.ethz.ch
19-945-351

The minimum for chaos in this curve is $\frac{\gamma}{\delta} > 14.76$, i.e. exactly the inverse of the previous result. Note that if we are exactly at this limit level no transverse zeros will be observed.

The heteroclinic tangle can be sketched as follows.

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