Nonlinear Dynamics and Chaos I. Assignment 5

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1 Hamiltonian System

Given the system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \mathbf{g}(x,y) = \begin{cases}\frac{\partial H(x,y)}{\partial y} + f_1(x,y)\\ -\frac{\partial H(x,y)}{\partial x} + f_2(x,y)\end{cases} \tag{1}$$

where H is twice continuosly differentiable and $\nabla \cdot \mathbf{f} = \nabla \cdot (f_1, f_2) \neq 0 \quad \forall (x,y) \in \mathbb{R}^2$.
We can show that in *any* simply connected domain $U \subset \mathbb{R}$ we have:

$$\nabla \cdot \mathbf{g} = \frac{\partial}{\partial x}\frac{\partial H(x,y)}{\partial y} - \frac{\partial}{\partial y}\frac{\partial H(x,y)}{\partial x} + \nabla \cdot \mathbf{f} = \nabla \cdot \mathbf{f} \neq 0 \quad \forall (x,y) \in U$$

Indeed this is true for the entire domain $\mathbb{R}^2$. We just used the general property of multi variable *differentiable* functions, guaranteeing that the differentiation order can be exchanged.

We can apply now Bendixson Criterion and conclude that there does not exist *any* limit cycle in $\mathbb{R}^2$ for the system given in eq. (1).

□ *q.e.d.*

2 Accuracy of Averaging

Given the system

$$\begin{cases}\dot{\mathbf{x}} = \varepsilon \mathbf{f}(\mathbf{x}, t, \varepsilon) & \mathbf{x} \in \mathbb{R}^n\\ \mathbf{x}(0) = \mathbf{x}_0\end{cases} \tag{2}$$

and its transformed system

$$\begin{cases}\dot{\mathbf{y}} = \varepsilon \bar{\mathbf{f}}(\mathbf{y}) & \mathbf{y} \in \mathbb{R}^n\\ \mathbf{y}(0) = \mathbf{x}_0 + \mathcal{O}(\varepsilon)\end{cases} \tag{3}$$

with the averaged system being $\bar{\mathbf{f}}(\mathbf{y}) = \frac{1}{T}\int_0^T \mathbf{f}(\mathbf{y}, t, 0)$, where T is the period of the smooth periodic function $\mathbf{f}$.

By integrating the right hand side of eqs. (2) and (3) we get:

$$\mathbf{x}(t) - \mathbf{x}_0 = \varepsilon \int_0^t \mathbf{f}(\mathbf{x}(\tau), \tau, \varepsilon) d\tau$$
$$\mathbf{y}(t) - (\mathbf{x}_0 + \mathcal{O}(\varepsilon)) = \varepsilon \int_0^t \bar{\mathbf{f}}(\mathbf{y}(\tau)) d\tau$$

Taking the difference of the two equations and putting the $\mathcal{O}(\varepsilon)$ term on the other side yields:

$$\mathbf{x}(t) - \mathbf{y}(t) = \varepsilon \int_0^t \left(\mathbf{f}(\mathbf{x}(\tau), \tau, \varepsilon) - \bar{\mathbf{f}}(\mathbf{y}(\tau))\right) d\tau + \mathcal{O}(\varepsilon) \tag{4}$$

In general we may write the function $\mathbf{f}$ as the sum of an oscillatory part an averaged part, i.e.:

$$\mathbf{f}(\mathbf{x}(t), t, \varepsilon) = \bar{\mathbf{f}}(\mathbf{x}(t)) + \tilde{\mathbf{f}}(\mathbf{x}(t), t, \varepsilon) \tag{5}$$

Substituting eq. (5) in eq. (4) while considering the absolute value, yields:

$$|\mathbf{x}(t) - \mathbf{y}(t)| = \varepsilon \int_0^t \left|\tilde{\mathbf{f}}(\mathbf{x}(\tau), \tau, \varepsilon) + \bar{\mathbf{f}}(\mathbf{x}(\tau)) - \bar{\mathbf{f}}(\mathbf{y}(\tau))\right| d\tau + \mathcal{O}(\varepsilon)$$

Which can then be upper estimated exploiting the fact that $\bar{\mathbf{f}}$ is Lipschitz and applying the triangle inequality:

$$|\mathbf{x}(t) - \mathbf{y}(t)| \leq \varepsilon \int_0^t \left|\tilde{\mathbf{f}}(\mathbf{x}(\tau), \tau, \varepsilon)\right| d\tau + \varepsilon L \int_0^t |\mathbf{x}(\tau) - \mathbf{y}(\tau)| d\tau + \mathcal{O}(\varepsilon) \tag{6}$$

where $L > 0$ is the chosen Lipschitz constant.

Further investigating the oscillatory part provides an upper estimate:

$$\left|\tilde{\mathbf{f}}(\mathbf{x}(\tau), \tau, \varepsilon)\right| \leq M, \qquad M := \max\left\{\left|\sup_{\tau\in[0,T]} \tilde{\mathbf{f}}(\mathbf{x}(\tau), \tau, \varepsilon)\right|, \left|\inf_{\tau\in[0,T]} \tilde{\mathbf{f}}(\mathbf{x}(\tau), \tau, \varepsilon)\right|\right\} \tag{7}$$

by simply exploiting the fact that $\tilde{\mathbf{f}}$ is a smooth, T-periodic function. Plugging eq. (7) into eq. (6) yields a further upper estimate:

$$\begin{aligned}|\mathbf{x}(t) - \mathbf{y}(t)| &\leq \varepsilon \int_0^t M d\tau + \varepsilon L \int_0^t |\mathbf{x}(\tau) - \mathbf{y}(\tau)| d\tau + \mathcal{O}(\varepsilon) = \\ &= \varepsilon M t + \varepsilon L \int_0^t |\mathbf{x}(\tau) - \mathbf{y}(\tau)| d\tau + \mathcal{O}(\varepsilon)\end{aligned} \tag{8}$$

This inequality is now in the form of the generalized Gronwall's inequality:

$$u(t) \leq c(t) + \int_0^t u(\tau) v(\tau) d\tau$$
$$\Rightarrow u(t) \leq c(0)\, e^{\int_0^t u(\tau) d\tau} + \int_0^t c'(\tau)\, e^{\int_\tau^t u(s) ds} d\tau$$

with the following nonnegative functions:

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$$v(t) = |\mathbf{x}(t) - \mathbf{y}(t)|$$
$$u(t) = \varepsilon M t$$
$$c(t) = \varepsilon L \qquad \Rightarrow c(0) = 0, \quad c'(t) = \varepsilon M$$

Applying those definitions to the Gronwall's inequality yields:

$$|\mathbf{x}(t) - \mathbf{y}(t)| \leq \int_0^t \varepsilon\, M\, e^{\int_\tau^t \varepsilon L ds} d\tau = \\ = \frac{M}{L}(e^{\varepsilon L t} - 1)$$

For values of $t \leq \frac{1}{\varepsilon}$ this expression can be upper estimated again to the constant term $C = \frac{M}{L}(e^{L} - 1)$ which is how we are able to conclude:

$$|\mathbf{x}(t) - \mathbf{y}(t)| \leq C + \mathcal{O}(\varepsilon) \qquad \forall\ t \leq 0 \leq \mathcal{O}(\frac{1}{\varepsilon}) \tag{9}$$

□ *q.e.d.*

3 Unsteady separation in time-periodic fluid flows

Consider the following time periodic fluid flow system:

$$\begin{cases} \dot{x} = u(x,y,t) & u(x,y,t) = u(x,y,t+T) \\ \dot{y} = v(x,y,t) & v(x,y,t) = v(x,y,t+T) \end{cases} \tag{10}$$

The following two assumptions are made:

$$u(x,0,t) = v(x,0,t) = 0 \qquad \text{no slip on wall} \tag{11}$$
$$u_x + v_y = 0 \qquad \text{incompressibility} \tag{12}$$

The no slip condition eq. (11) guarantees that we have fixed points along the wall in any point $\mathbf{p}_0 = (x_0, 0)$, which turn out to be non-hyperbolic, given that $\det J = u_x v_y - u_y u_x = -v_y^2 + v_y^2 = 0$, using the incompressibility condition eq. (12).

Given that in $y = 0$ the velocity fields evaluate both to 0, we may write[1]:

$$u(x,y,t) = u(x,y,t) - u(x,0,t) = u(x,sy,t)\big|_{s=0}^{1} = \\ = \int_0^1 \frac{d}{ds} u(x,sy,t) ds = \int_0^1 u_y(x,sy,t)\, y\, ds = y \int_0^1 u_y(x,sy,t)\, ds \tag{13}$$

Equation (13) applies in a completely analogous fashion to v as well. This yields:

$$\begin{cases} \dot{x} &= y \int_0^1 u_y(x,sy,t) ds \\ \dot{y} &= y \int_0^1 v_y(x,sy,t) ds \end{cases} \tag{14}$$

[1]Reusing the fact that u and v are differentiable

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Furthermore, the no slip condition also forces the derivative of the velocity along x in the wall to be zero, meaning $u_x(x, 0, t) = 0$. From the incompressibility condition eq. (12) it follows that $v_y(x, 0, t) = 0$. With this property we can proceed in an analogous fashion to eq. (13):

$$\begin{aligned} v_y(x, sy, t) &= v_y(x, sy, t) - v_y(x, 0, t) = v_y(x, psy, t)\Big|_{p=0}^{1} = \\ &= \int_0^1 \frac{d}{dp} v_y(x, psy, t) dp = \int_0^1 v_{yy}(x, psy, t)\, sy\, dp = y \int_0^1 v_{yy}(x, psy, t)\, s\, dp \end{aligned} \tag{15}$$

This yields:

$$\begin{cases} \dot{x} &= y \int_0^1 u_y(x, sy, t) ds \\ \dot{y} &= y^2 \int_0^1 \int_0^1 v_{yy}(x, spy, t) p\, dp\, ds \end{cases} \tag{16}$$

To scaling the equations to a region near the wall we set $y = \varepsilon\eta$, which yields:

$$\begin{cases} \dot{x} &= \varepsilon\, \eta \int_0^1 u_y(x, s\varepsilon\eta, t) ds \\ \dot{\eta} &= \varepsilon\, \eta^2 \int_0^1 \int_0^1 v_{yy}(x, sp\varepsilon\eta, t) p\, dp\, ds \end{cases} \tag{17}$$

Since we are interested in the case where $0 \leq \varepsilon << 1$, eq. (17) is in a suitable form for *averaging*. From the lecture together with the proof provided in section 2, we know that a coordinate transformation exists involving the oscillatory part (see eq. (5) for reference), for which solutions of the averaged system remain ε close for $\mathcal{O}(\frac{1}{\varepsilon})$ time. The averaged system is thereby calculated over one period T with $\varepsilon = 0$, which allows us to solve the integrals by placing the now independent part outside, and solving the definite integrals yielding simply 1 and $\frac{1}{2}$ respectively. In this way we get, by emphasizing the fact of having averaged equations now with a bar on the differential equation variables:

$$\begin{cases} \dot{\bar{x}} &= \varepsilon \int_0^T \bar{\eta}\, u_y(\bar{x}, 0, t') dt' \\ \dot{\bar{\eta}} &= \varepsilon \int_0^T \frac{1}{2} \bar{\eta}^2 v_{yy}(\bar{x}, 0, t') dt' \end{cases} \tag{18}$$

In this way, the right hand side of eq. (18) becomes *independent* of time, i.e. we now have an autonomous system. To simplify the equations we rescale time on the averaging integrals to be:

$$\eta(t) = \frac{d\tau}{dt} \tag{19}$$

and reformulate the equations in the new time τ:

$$\begin{cases} \bar{x}' &= \varepsilon \int_0^{T'} u_y(\bar{x}, 0, \tau') d\tau' \\ \bar{\eta}' &= \varepsilon \int_0^{T'} \frac{1}{2} \bar{\eta}\, v_{yy}(\bar{x}, 0, \tau') d\tau' \end{cases} \tag{20}$$

The Jacobian of this system can be shown to be:

$$J(x, \eta)\big|_{(x,\eta)=(x_0,0)} = \begin{pmatrix} \varepsilon \int_0^{T'} u_{yx}(\bar{x}, 0, \tau') d\tau' & 0 \\ 0 & \varepsilon \int_0^{T'} \frac{1}{2} \bar{v}_{yy}(\bar{x}, 0, \tau') d\tau' \end{pmatrix} \tag{21}$$

Deriving the incompressibility condition from eq. (12) in y direction, we can transform this into:

$$J(x, \eta)\big|_{(x,\eta)=(x_0,0)} = \begin{pmatrix} -\varepsilon \int_0^{T'} v_{yy}(\bar{x}, 0, \tau') d\tau' & 0 \\ 0 & \varepsilon \int_0^{T'} \frac{1}{2} \bar{v}_{yy}(\bar{x}, 0, \tau') d\tau' \end{pmatrix} \tag{22}$$

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Which directly yields the two eigenvalues $-\lambda,\ +\frac{\lambda}{2}$. So in order for the fixed point to have an unstable manifold, we require $\frac{\lambda}{2} > 0$, i.e.:

$$\varepsilon \int_0^{T'} \frac{1}{2}\bar{v}_{yy}(\bar{x}, 0, \tau')d\tau' > 0 \tag{23}$$

Furthermore we impose to have a fixed point in eq. (20):

$$\varepsilon \int_0^{T'} u_y(\bar{x}, 0, \tau')d\tau' = 0 \tag{24}$$

Thanks to the averaging theorem, eqs. (23) and (24) can be related to the original system. Transforming them back in the originally scaled time and to the global system in y, i.e. without ε vicinity to the wall, we finally get:

$$\int_0^{T} u_y(x, 0, t)dt = 0 \tag{25}$$

$$\int_0^{T} v_{yy}(x, 0, t)dt > 0 \tag{26}$$

□ $q.e.d.$

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