ETH zürich IMES

Nonlinear Dynamics and Chaos I. Assignment 4

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1 Discrete dynamical system

The following system is given:

$$\begin{aligned} \mathbf{x}_{n+1} &= A\mathbf{x}_n + \mathbf{f}(\mathbf{x}_n, \mathbf{y}_n) \qquad \mathbf{x}_n \in \mathbb{R}^c \quad A \in \mathbb{R}^{c\times c} \\ \mathbf{y}_{n+1} &= B\mathbf{y}_n + \mathbf{g}(\mathbf{x}_n, \mathbf{y}_n) \qquad \mathbf{y}_n \in \mathbb{R}^d \quad B \in \mathbb{R}^{d\times d} \quad \mathbf{f}, \mathbf{g} \in C^r \text{ only nonlinear terms} \end{aligned} \tag{1}$$

Given the assumption that all the eigenvalues of A have modulus one, and none of the eigenvalues of B have modulus one, we already are in coordinates for which the behavior in x is reflected in the center subspace, an the behavior in y is reflected in the stable / unstable subspaces for the linearized system at the origin.

1.1 Algebraic equation for the center manifold

By construction, the linearized system is already in the block-diagonalized form as it was the case for the continuous systems:

$$\begin{pmatrix} \mathbf{x}_{n+1} \\ \mathbf{y}_{n+1} \end{pmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{pmatrix} \mathbf{x}_n \\ \mathbf{y}_n \end{pmatrix} + \begin{pmatrix} f(\mathbf{x}_n, \mathbf{y}_n) \\ g(\mathbf{x}_n, \mathbf{y}_n) \end{pmatrix} \tag{2}$$

Knowing the existence of the center manifold, by the center manifold theorem analogous to the continuous cases, in the given coordinates we can construct the function:

$$\mathbf{y}(k) = \mathbf{h}(\mathbf{x}(k)) \qquad \mathbf{h} : \mathbb{R}^c \to \mathbb{R}^d \quad \mathbf{h} \in C^{r-1} \tag{3}$$

In an analogous way we may derive the function, to obtain a difference equation:

$$\begin{aligned} \Delta\mathbf{y}(k) &= \mathrm{D}\,\mathbf{h}(\mathbf{x})\,\Delta\mathbf{x}(k) \\ &= \mathrm{D}\,\mathbf{h}(\mathbf{x})\left[(A - \mathbb{I}_c)\,\mathbf{x}_k + \mathbf{f}\,(\mathbf{x}_k, \mathbf{y}_k)\right] \\ &= (B - \mathbb{I}_d)\,\mathbf{y}_k + \mathbf{g}(\mathbf{x}_k, \mathbf{y}_k) \end{aligned} \tag{4}$$

where the difference equations have been obtained by simply subtracting $\mathbf{x}_k$ and $\mathbf{y}_k$ respectively from eq. (1), which is why the identity matrices are appearing in their respective sizes.

We may now Taylor expand $\mathbf{h}(\mathbf{x})$ in the fixed point $\mathbf{x} = 0$:

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}(0) + \mathrm{D}\,\mathbf{h}(0)\,\mathbf{x} + \frac{1}{2}\mathrm{D}^2\,\mathbf{h}(\mathbf{x}) \otimes \mathbf{x} \otimes \mathbf{x} + \mathcal{O}\left(|\mathbf{x}|^3\right) \tag{5}$$

By definition of fixed point, the manifold needs to pass through the origin, and given that they are linearly independent there cannot be a linear dependence which is why we get:

- $\mathbf{h}(0) = 0$
- $\mathrm{D}\,\mathbf{h}(0) = 0$

Therefore, without loss of generality we may assume $\mathbf{h}(\mathbf{x})$ to be in the form of:

$$\mathbf{h}(\mathbf{x}) = C \otimes \mathbf{x} \otimes \mathbf{x} + \mathcal{O}\left(|\mathbf{x}|^3\right) \tag{6}$$

where C is a third order tensor, whose components can be determined by plugging eq. (6) into the results obtained before in eq. (4), while expressing $\mathbf{y}_k$ as $\mathbf{h}(\mathbf{x}_k)$ and setting $\mathbf{y}_k = \mathbf{y}$ and $\mathbf{x}_k = \mathbf{x}$ i.e.:

$$\mathrm{D}\,\mathbf{h}(\mathbf{x})\left[(A - \mathbb{I}_c)\,\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x}))\right] = (B - \mathbb{I}_d)\,\mathbf{h}(\mathbf{x}) + \mathbf{g}(\mathbf{x}, \mathbf{h}(\mathbf{x})) \tag{7}$$

A coefficient comparison will determine all the unknown components of C. However, given that the center manifold has a deficiency in its differentiability and is not guaranteed to exist uniquely, generally the *Taylor series may not converge*.

1.2 Discrete example system

Analyzing now the system

$$\begin{aligned} x_{n+1} &= x_n + x_n y_n \\ y_{n+1} &= \lambda y_n - x_n^2 \qquad \lambda \in (0,1) \end{aligned} \tag{8}$$

We can simply observe that this is exactly in the same form as proposed in eq. (1) with the following parameters:

$A = 1$ $\qquad f(x_n, y_n) = x_n\,y_n$

$B = \lambda$ $\qquad g(x_n, y_n) = -x_n^2$

Note that the eigenvalue of a scalar value is the value itself, which shows also the required conditions from eq. (1) to hold, namely $A = 1$ and $B \neq 1$.

Given that in this case the subspaces have dimension 1, we can write $h(x)$ in the following trivial form:

$$h(x) = c\,x^2 + d\,x^3 + \mathcal{O}(x^4) \tag{9}$$

Plugging this concrete example into the previously obtained general eq. (7) we get:

$$\begin{aligned} \left(2\,c\,x + 3\,d\,x^2\right)\left(c\,x^3 + d\,x^4\right) &= (\lambda - 1)\left(c\,x^2 + d\,x^3\right) - x^2 \\ \mathcal{O}(x^4) &= ((\lambda - 1)\;c - 1)\;x^2 + (\lambda - 1)\;d\,x^3 \end{aligned}$$

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From the coefficient comparison it follows that

$$c = -\frac{1}{1-\lambda} \qquad d = 0 \tag{10}$$

We therefore have as a result for the center manifold:

$$h(x) = -\frac{1}{1-\lambda} x^2 \tag{11}$$

Together with some simulated trajectories, this is shown in fig. 1.

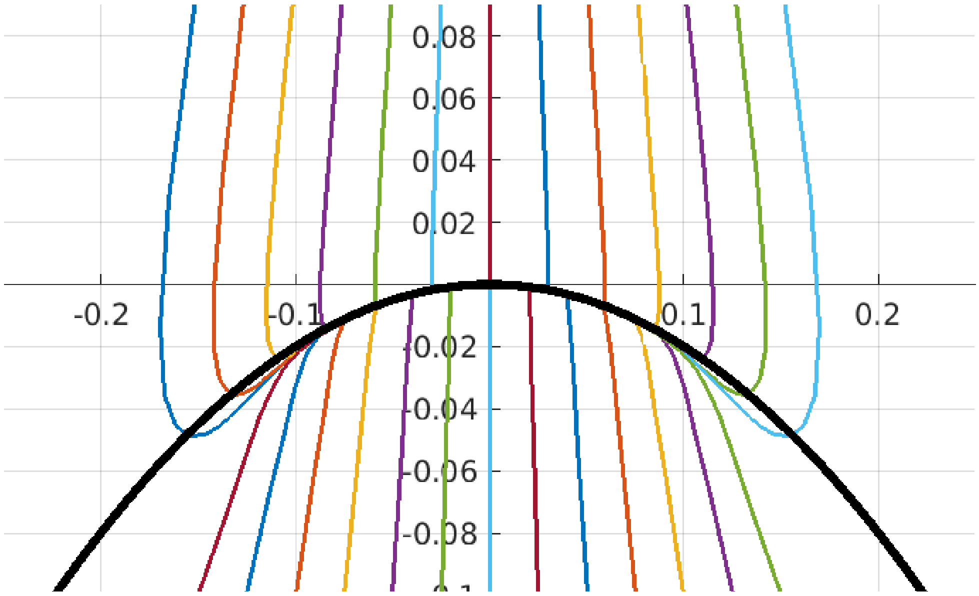

Figure 1: Center manifold for discrete example system

The reduced system dynamics can be written as:

$$x_{n+1} = x_n + x_n\, h(x_n) = x_n - \frac{1}{1-\lambda} x_n^3 \tag{12}$$

So near the fixed point 0, the change with respect to progressing time steps is:

$$\Delta\, x_n = x_{n+1} - x_n = -\frac{1}{1-\lambda} x_n^3 \tag{13}$$

Note that on this reduced dynamics we have:

$$\forall\ x > 0 \quad \Delta\, x_n < 0 \qquad \forall\ x < 0 \quad \Delta\, x_n > 0 \tag{14}$$

In one dimension this is indeed enough to conclude asymptotic stability, meaning that $\Delta x(x)$ just needs to be a strictly monotonically decreasing function.[1] We therefore expect the fixed point $x = 0$ to be asymptotically stable.

Indeed, numerical simulation shows the expected behavior given in fig. 2.

[1]In more dimension this could not be done, because inequalities can only be applied to scalars

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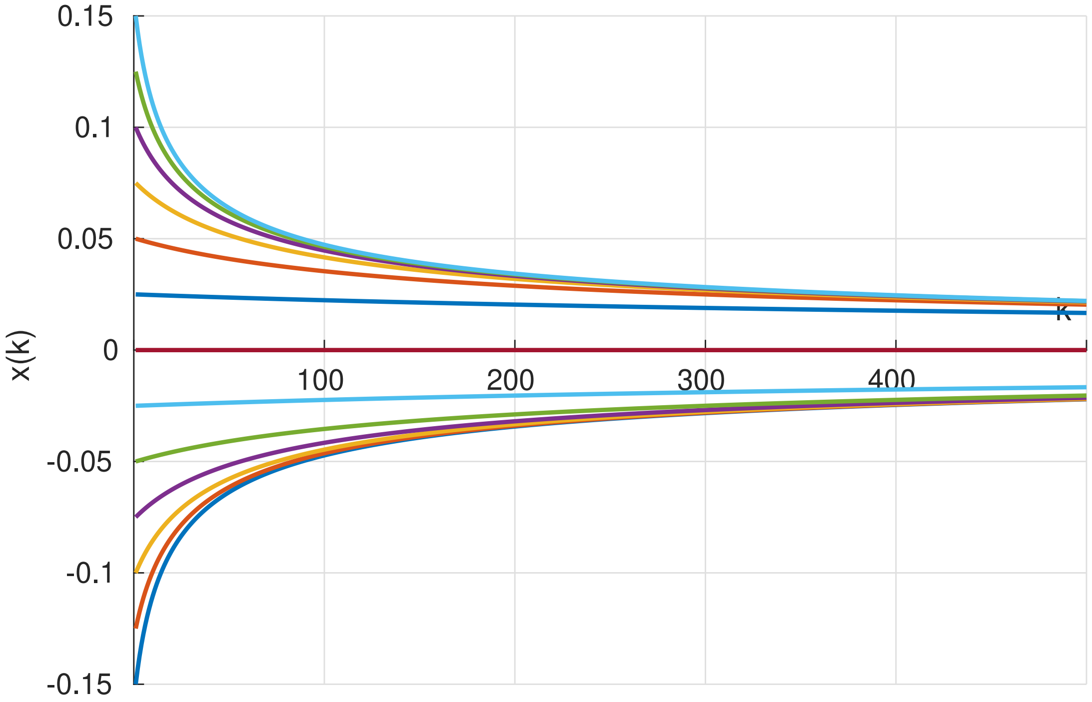

Figure 2: Reduced dynamics simulation

2 Duffing Equation

Consider the following system

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= \beta u - u^2 - \delta v \end{aligned} \tag{15}$$

To analyze the β dependent center manifold, we extend the system in the following way:

$$\tilde{\mathbf{u}} = \begin{pmatrix} u \\ \beta \end{pmatrix} \tag{16}$$

$$\dot{\tilde{\mathbf{u}}} = \begin{pmatrix} \dot{u} \\ 0 \end{pmatrix} \tag{17}$$

$$\Rightarrow \begin{pmatrix} \dot{\tilde{\mathbf{u}}} \\ \dot{v} \end{pmatrix} = \mathbf{F}(\tilde{\mathbf{u}}, v) = \begin{cases} \dot{u} & = v \\ \dot{\beta} & = 0 \\ \dot{v} & = \beta u - u^2 - \delta v \end{cases} \tag{18}$$

By linearizing the extended system around the origin, we find the following matrix:

$$\mathrm{D}\,\mathbf{F}(0) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \beta - 2u & u & -\delta \end{bmatrix}\Bigg|_{(u,\beta,v)=(0,0,0)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -\delta \end{bmatrix} \tag{19}$$

It can be seen immediately that in the origin (with $\beta = 0$) we have a center subspace of dimension 2, given that there are two zero eigenvalues. The stable subspace, constituted by the

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eigenvector related to the eigenvalue $\lambda_3 = -\delta$, has dimension one. It is linearly independent with respect to the plane of center subspace of the directions in β and u. Therefore, we may express the center manifold in vicinity to the origin as:

$$v = h(\tilde{\mathbf{u}}) = h(u, \beta) = a\,u^2 + b\,u\,\beta + c\,\beta^2 + \mathcal{O}(3) \tag{20}$$

Where $\mathcal{O}(3)$ indicates all possible third order terms, namely $\mathcal{O}\left(|\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}|\right)$. Derivation in time now yields:

$$\begin{aligned}
\dot{v} &= \mathrm{D}\,h(\tilde{\mathbf{u}}) \begin{pmatrix} \dot{u} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} 2\,a\,u + b\,\beta & b\,u + 2\,c \end{pmatrix} \begin{pmatrix} \dot{u} \\ 0 \end{pmatrix} = \\
&= (2\,a\,u + b\,\beta)\,\dot{u} = (2\,a\,u + b\,\beta)\,v \\
&= (2\,a\,u + b\,\beta)\left(a\,u^2 + b\,u\,\beta + c\,\beta^2 + \mathcal{O}(3)\right) \\
&= \mathcal{O}(3)
\end{aligned} \tag{21}$$

On the other hand, we can express $\dot{v}$ by simply using the initial equation eq. (15):

$$\begin{aligned}
\dot{v} &= \beta\,u - u^2 - \delta\,v = \beta\,u - u^2 - \delta\,h(\tilde{\mathbf{u}}) \\
&= \beta\,u - u^2 - \delta\,\left(a\,u^2 + b\,u\,\beta + c\,\beta^2 + \mathcal{O}(3)\right) \\
&= u^2\,(-\delta\,a - 1) + (1 - \delta\,b)\,u\,\beta - \delta\,c\,\beta^2
\end{aligned} \tag{22}$$

Putting eqs. (21) and (22) together, we get for a third order truncation:

$$u^2\,(-\delta\,a - 1) + (1 - \delta\,b)\,u\,\beta - \delta\,c\,\beta^2 = 0 \tag{23}$$

Comparing the coefficients, to satisfy the equation we need:

$$a = -\frac{1}{\delta} \qquad b = \frac{1}{\delta} \qquad c = 0$$

By back-substitution, we finally obtain:

$$v = h(u, \beta) = -\frac{1}{\delta}\,u^2 + \frac{1}{\delta}\,u\,\beta + \mathcal{O}(3) \tag{24}$$

With this relationship we get the reduced system on the center manifold:

$$\dot{u} = h(u, \beta) = -\frac{1}{\delta}\,u^2 + \frac{1}{\delta}\,u\,\beta + \mathcal{O}(3) \tag{25}$$

We can observe that this is in the form of universal unfolding:

$$\dot{u} = \frac{1}{\delta}\,u\,(\beta - u) \tag{26}$$

Constructing a stability diagram of this reduced system, while carefully taking into account all sign changes, leads to a sketch as shown in fig. 3.

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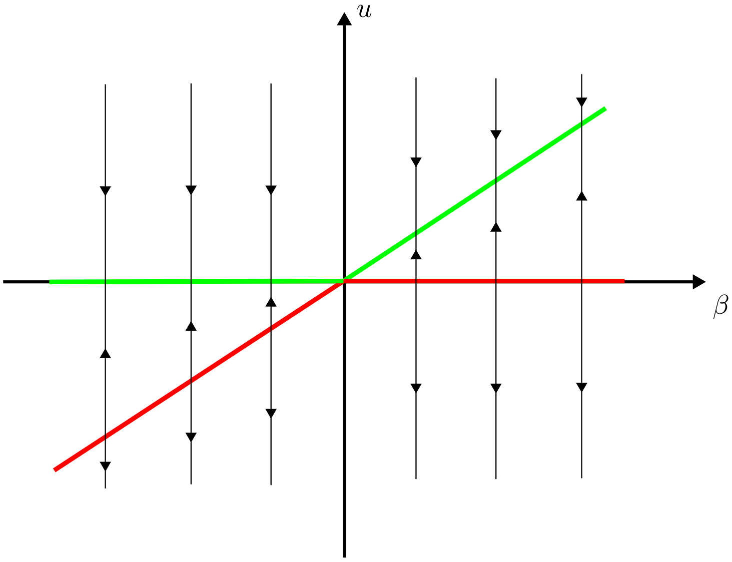

Figure 3: Stability diagram of reduced duffing equation, showing a transcritical bifurcation

3 Pendulum's unstable manifold approximation

Given the pendulum equation:

$$\ddot{x} + \sin x = 0 \tag{27}$$

In state space this becomes:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \mathbf{f}(\mathbf{x}) = \begin{cases} x_2 \\ -\sin x_1 \end{cases} \qquad \text{with } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \tag{28}$$

To linearize the system, we calculate:

$$\mathrm{D}\,\mathbf{f}(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & 0 \end{bmatrix} \tag{29}$$

We can see that the fixed point $(0,0)$ has purely imaginary eigenvalues, given that $\lambda^2 + 1 = 0$ does not offer real solutions. We therefore select the hyperbolic fixed point $\mathbf{p} = (\pm\pi, 0)$:

$$\mathrm{D}\,\mathbf{f}(\mathbf{p}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with the following eigenvalue / eigenvector pairs:

$$\mathbf{s}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_1 = 1$$
$$\mathbf{s}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \lambda_2 = -1$$

This means that in $\mathbf{p}$ we can define a stable subspace E^s as a span of the eigenvector $\mathbf{s}_2$ related to the stable eigenvalue λ_2 and an unstable subspace E^u related to $\mathbf{s}_1$. Since there is no center

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subspace, i.e. $c = 0$, the center manifold theorem guarantees the existence of unique $W^u(\mathbf{p})$ and $W^s(\mathbf{p})$ manifolds, which in $\mathbf{p}$ are *tangent* to E^u and E^s respectively.

To construct a local approximation for the unstable manifold W^s, we apply a change of coordinates to be in the eigenvector space. In this way stable and unstable manifolds will be separable:

$$\mathbf{x} = T\,\xi = \begin{bmatrix}\mathbf{s}_1 & \mathbf{s}_2\end{bmatrix}\xi = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\xi$$
$$\xi = T^{-1}\,\mathbf{x}$$

Given that T is orthonormal, using $T^{-1} = T^{\top}$, it easily follows:

$$\xi = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\mathbf{x} \tag{30}$$

To get the differential equation in this coordinate system, we calculate:

$$\dot{\xi} = T^{-1}\,\dot{\mathbf{x}} = T^{-1}\,\mathbf{f}(\mathbf{x}) = T^{-1}\,\mathbf{f}(T\,\xi) \tag{31}$$

Before Taylor-expanding $\mathbf{f}(\mathbf{x})$, we introduce $\tilde{\mathbf{f}}(\mathbf{x})$, which translates the fixed point $\mathbf{p} = \begin{pmatrix}\pm\pi\\ 0\end{pmatrix}$ to the origin:

$$\tilde{\mathbf{f}}(\mathbf{x}) = \mathbf{f}(x_1 \pm \pi, x_2) = \begin{cases}x_2\\ -\sin(x_1 \pm \pi)\end{cases} = \begin{cases}x_2\\ \sin x_1\end{cases} \tag{32}$$

In this way the linearizion becomes:

$$\mathrm{D}\,\tilde{\mathbf{f}}(\mathbf{x}) = \left.\begin{bmatrix}0 & 1\\ \cos x_1 & 0\end{bmatrix}\right|_{\mathrm{x}=(0,0)} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \tag{33}$$

which is equivalent to the results obtained before. The following results regarding $\tilde{\mathbf{f}}$ around $p = (0,0)$ can be naturally transfered to $\mathbf{f}$ in $\mathbf{p} = (\pm\pi, 0)$.

In this way eq. (31) becomes:

$$\dot{\xi} = T^{-1}\,\tilde{\mathbf{f}}(T\,\xi) \tag{34}$$

We can now conveniently Taylor-expand $\tilde{\mathbf{f}}$ around $\mathbf{p} = (0,0)$, yielding:

$$\tilde{\mathbf{f}}(\mathbf{x}) = \tilde{\mathbf{f}}(\mathbf{p}) + \mathrm{D}\,\tilde{\mathbf{f}}(\mathbf{p})(\mathbf{x}) + \frac{1}{2}\mathrm{D}^2\,\tilde{\mathbf{f}}(\mathbf{p}) \otimes \mathbf{x} \otimes \mathbf{x} + \mathcal{O}\left(|\mathbf{x}|^3\right) \tag{35}$$

By definition of fixed point, $\tilde{\mathbf{f}}(\mathbf{p}) = 0$. Furthermore, we can apply the transformation, to write eq. (34) completely in the transformed coordinates, this yields:

$$\dot{\xi} = T^{-1}\left(\mathrm{D}\,\tilde{\mathbf{f}}(\mathbf{p})(T\,\xi) + \frac{1}{2}\mathrm{D}^2\,\tilde{\mathbf{f}}(\mathbf{p}) \otimes (T\,\xi) \otimes (T\,\xi) + \mathcal{O}\left(|T\,\xi|^3\right)\right) \tag{36}$$

We first calculate $T\,\xi$:

$$T\,\xi = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\xi = \tfrac{1}{\sqrt{2}}\begin{pmatrix}u - v\\ u + v\end{pmatrix} \tag{37}$$

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with

$$\xi = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = T\,\xi = \tfrac{1}{\sqrt{2}} \begin{pmatrix} u - v \\ u + v \end{pmatrix} \tag{38}$$

The linear term yields:

$$\begin{aligned} T^{-1}\mathrm{D}\,\tilde{\mathbf{f}}(\mathbf{p}) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} &= \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} \tilde{v} \\ \tilde{u} \end{pmatrix} \\ &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} \tilde{v} + \tilde{u} \\ -\tilde{v} + \tilde{u} \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} u + v + u - v \\ -u - v + u - v \end{pmatrix} \\ &= \begin{pmatrix} u \\ -v \end{pmatrix} \end{aligned}$$

The second order derivative, becomes a 3rd order tensor, since we need to derive the Jacobian in 2 directions, namely x_1 and x_2:

$$\mathrm{D}^2\,\tilde{\mathbf{f}}(\mathbf{p}) = \mathrm{D}(\mathrm{D}\,\tilde{\mathbf{f}}(\mathbf{x}))\big|_{\mathbf{x}=\mathbf{p}}$$

$$\mathrm{D}_{x_1}(\mathrm{D}\,\tilde{\mathbf{f}}(\mathbf{x}))\big|_{\mathbf{x}=\mathbf{p}} = \mathrm{D}_{x_1} \begin{bmatrix} 0 & 1 \\ \cos x_1 & 0 \end{bmatrix}\Bigg|_{\mathbf{x}=(0,0)} = \begin{bmatrix} 0 & 0 \\ -\sin x_1 & 0 \end{bmatrix}\Bigg|_{\mathbf{x}=(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{39}$$

$$\mathrm{D}_{x_2}(\mathrm{D}\,\tilde{\mathbf{f}}(\mathbf{x}))\big|_{\mathbf{x}=\mathbf{p}} = \mathrm{D}_{x_2} \begin{bmatrix} 0 & 1 \\ \cos x_1 & 0 \end{bmatrix}\Bigg|_{\mathbf{x}=(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{40}$$

As we can observe, the second derivative completely vanishes. This means that we need to take into account also the 3rd order term, being:

$$T^{-1}\left(\frac{1}{6}\mathrm{D}^3\,\tilde{\mathbf{f}}(\mathbf{p}) \otimes (T\,\xi) \otimes (T\,\xi) \otimes (T\,\xi)\right) \tag{41}$$

Since the 2nd order derivative given in eq. (40) was zero and eq. (39) only depends on x_1, the only non-zero derivative direction for the 3rd order being represented by the 4th order tensor $\mathrm{D}^3\,\tilde{\mathbf{f}}(\mathbf{p})$, will be x_1, $x_1\,x_1$:

$$\mathrm{D}_{x_1,x_1}(\mathrm{D}\,\tilde{\mathbf{f}}(\mathbf{x}))\big|_{\mathbf{x}=\mathbf{p}} = \mathrm{D}_{x_1} \begin{bmatrix} 0 & 0 \\ -\sin x_1 & 0 \end{bmatrix}\Bigg|_{\mathbf{x}=(0,0)} = \begin{bmatrix} 0 & 0 \\ -\cos x_1 & 0 \end{bmatrix}\Bigg|_{\mathbf{x}=(0,0)} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \tag{42}$$

If we look at the other part of the outer product, while denoting $T\,\xi$ as $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$, we can observe that

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \otimes \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \otimes \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

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is a third order tensor, which we may represent as 2 matrices:

$$\begin{bmatrix} \tilde{u}^3 & \tilde{u}^2\,\tilde{v} \\ \tilde{u}^2\,\tilde{v} & \tilde{v}^3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{u}^2\,\tilde{v} & \tilde{u}\,\tilde{v}^2 \\ \tilde{u}\,\tilde{v}^2 & \tilde{v}^3 \end{bmatrix}$$

Another outer-product multiplication with the matrix found in eq. (42), yields then (for the second row) the following third order terms:

$$-\tilde{u}^3 - 2\,\tilde{u}^2\,\tilde{v} - \tilde{u}\,\tilde{v}^2 \tag{43}$$

When back-substituting the definitions of $\tilde{u}$ and $\tilde{v}$ given in eq. (38), after some simplifications we get:

$$-\sqrt{2}\,u^2(u-v)$$

By continuing in the multiplication chain of eq. (41), we need to apply the transformation as given in eq. (30), we get:

$$-\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\frac{1}{6}\sqrt{2}\,u^2(u-v) = -\frac{1}{6}\begin{bmatrix}1\\1\end{bmatrix}u^2(u-v) \tag{44}$$

Putting these new findings together with the linear terms, we find the equation eq. (28) in our new reference frame to be [2]:

$$\dot{\xi} = \begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{cases} u - \frac{1}{6}\,u^2(u-v) + \mathcal{O}\left(\left(u^2+v^2\right)^2\right) \\ -v - \frac{1}{6}\,u^2(u-v) + \mathcal{O}\left(\left(u^2+v^2\right)^2\right) \end{cases} \tag{45}$$

The following steps consider the 3rd order truncated version of eq. (45). To determine the unstable manifold we need to find $v = h(u)$. By deriving this function in time we get another way of expressing $\dot{v}$, which can be exploited in the following way:

$$\begin{aligned} \dot{v} &= \mathrm{D}\,h(u)\,\dot{u} = \mathrm{D}\,h(u)\,\left(u - \tfrac{1}{6}\,u^2(u-h(v))\right) = \\ &= \dot{v} = -h(u) - \tfrac{1}{6}\,u^2(u-h(u)) \end{aligned} \tag{46}$$

Without loss of generality we can assume $h(u)$ to be in the form of:

$$h(u) = a\,u^2 + b\,u^3 + \mathcal{O}(u^4) \tag{47}$$

Substituting this in eq. (46) yields:

$$\begin{aligned} 2\,a\,u^2 + 3\,b\,u^3 + \mathcal{O}(u^4) &= -a\,u^2 + b\,u^3 - \tfrac{1}{6}\,u^2(u - a\,u^2 + b\,u^3) \\ &= -a\,u^2 + (b - \tfrac{1}{6})\,u^3 + \mathcal{O}(u^4) \end{aligned}$$

Comparing the coefficients we get:

[2] Note that the order term $\mathcal{O}\left(|T\,\xi|^4\right)$ can be simplified to the form shown in eq. (45).

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$$\begin{cases} 2\,a = -a \\ 3\,b = b - \frac{1}{6} \end{cases}$$

Which yields:

$$a = 0 \qquad b = -\frac{1}{12} \tag{48}$$

So finally we can express $h(u)$ as:

$$v = h(u) = -\frac{1}{12}\,u^3 \tag{49}$$

Note that as a third order approximation, this result is unique, as guaranteed by the center manifold theorem. Figure 4 shows a the result graphically, back-transformed in the original system coordinates of $\tilde{\mathbf{f}}(\mathbf{x})$, i.e. still centered in $\mathbf{p}$.

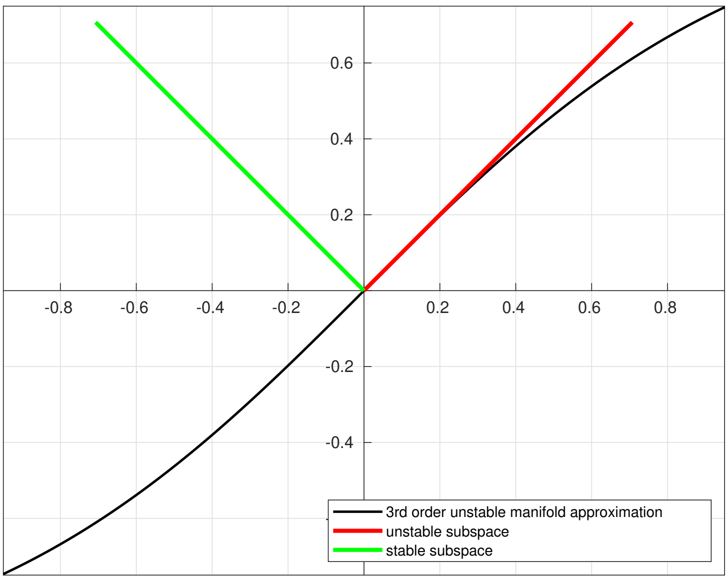

Figure 4: Unstable manifold approximation plot of $\tilde{\mathbf{f}}$

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