Nonlinear Dynamics and Chaos I. Assignment 3

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1 Lorenz Equations

This section treats the Lorenz equations as given in eq. (1).

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = b\,x - y - x\,z \\ \dot{z} = x\,y - c\,z \end{cases} \qquad a,\, b,\, c > 0; \quad a > 1 + c \tag{1}$$

1.1 Unstability of all equilibria

To analyze under which conditions all possible equilibria become unstable, first we find them by solving[1]:

$$\mathbf{f}(x^\star, y^\star, z^\star) = \mathbf{0} \tag{2}$$

The trivial solution can by found immediately to be:

$$\begin{pmatrix} x_1^\star \\ y_1^\star \\ z_1^\star \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

The solutions are all found by considering $x = y$[2], from which the second equation yields $z = b - 1$ and the last equation finally $x = \overset{+}{\underset{(-)}{}}\sqrt{c(b-1)}$, in summary:

$$\begin{pmatrix} x_{2/3}^\star \\ y_{2/3}^\star \\ z_{2/3}^\star \end{pmatrix} = \begin{pmatrix} \overset{+}{\underset{(-)}{}}\sqrt{c\,(b-1)} \\ \overset{+}{\underset{(-)}{}}\sqrt{c\,(b-1)} \\ b - 1 \end{pmatrix} \tag{4}$$

[1] considering eq. (1) as $\begin{pmatrix} \dot{x} & \dot{y} & \dot{z} \end{pmatrix}^\top = \mathbf{f}(x, y, z)$

[2] only alternative from the first equation in eq. (1)

1.1.1 Stability Analysis

To determine the stability of these fixed points, we firstly need to linearize the system, by computing its Jacobian:

$$\mathbf{J}(x,y,z) = \begin{bmatrix} -a & a & 0 \\ b-z & -1 & -x \\ y & x & -c \end{bmatrix} \tag{5}$$

Substitution of the fixed points given in eqs. (3) and (4) yields:

$$\mathbf{J}(x_1^\star, y_1^\star, z_1^\star) = \begin{bmatrix} -a & a & 0 \\ b & -1 & 0 \\ 0 & 0 & -c \end{bmatrix}$$
$$\mathbf{J}(x_{2/3}^\star, y_{2/3}^\star, z_{2/3}^\star) = \begin{bmatrix} -a & a & 0 \\ 1 & -1 & \overset{(+)}{-}\sqrt{c(b-1)} \\ \overset{(-)}{+}\sqrt{c(b-1)} & \overset{(-)}{+}\sqrt{c(b-1)} & -c \end{bmatrix}$$

Then the characteristic polynomial can be derived by:

$$\det\left(\mathbf{J}(x_1^\star, y_1^\star, z_1^\star) - \lambda\mathbb{I}\right) = 0$$
$$\det\left(\mathbf{J}(x_{2/3}^\star, y_{2/3}^\star, z_{2/3}^\star) - \lambda\mathbb{I}\right) = 0$$

After some calculations we get:

$$-\lambda^3 - (a+c+1)\lambda^2 + (ab - ac - a - c)\lambda + abc - ac = 0$$
$$-\lambda^3 - (a+c+1)\lambda^2 - (ac+bc)\lambda + 2ac - 2abc = 0$$

As we may observe, the sign of the zero-order terms depends on b. Starting from the assumption to prove we get:

$$b > \frac{a(3+a+c)}{a-c-1} > \frac{a(3+a+c)}{a} > 3+a+c > 1 \qquad \text{for } a,b,c > 0;\ a > c+1$$

We note that the first equation does not satisfy the necessary conditions $a_i > 0$, i.e. all coefficients need to be positive having $a_3 = -1$ for $b > 1$. In this way the trivial fixed point turns out to be unstable. For $b > 1$ the second equation needs to be multiplied by -1, to be able to apply Ruth-Horwitz:

$$\lambda^3 + (a+c+1)\lambda^2 + (ac+bc)\lambda + 2abc - 2ac = 0 \tag{6}$$

The Ruth-Horwitz determinants are as follows:

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$$D_0 = a_0 = 2abc - 2ac > 0$$
$$D_1 = a_1 = ac + bc > 0$$
$$D_2 = \det \begin{pmatrix} ac + bc & 2abc - 2ac \\ 1 & a + c + 1 \end{pmatrix} = c\left(a^2 + a(c + 3 - b) + b(c + 1)\right)$$
$$D_3 = a_3 D_2 = D_2$$

It results that the only chance to get instability is $D_2 < 0$:

$$\begin{aligned} c\left(a^2 + a(c + 3 - b) + b(c + 1)\right) &< 0 \\ a^2 + ac + 3a - ab + bc + b &< 0 \\ a\,(a + c + 3) + b\,(c + 1 - a) &< 0 \\ a\,(a + c + 3) &< b\,(a - c - 1) \\ b &> \frac{a\,(a + c + 3)}{a - c - 1} \end{aligned}$$

□ *q.e.d.*

1.2 Numerical trajectory

Setting the parameters to $a = 10$, $b = 28$, and $c = \frac{8}{3}$, the trajectories evolve around the unstable fixed point in the origin as shown in fig. 1.

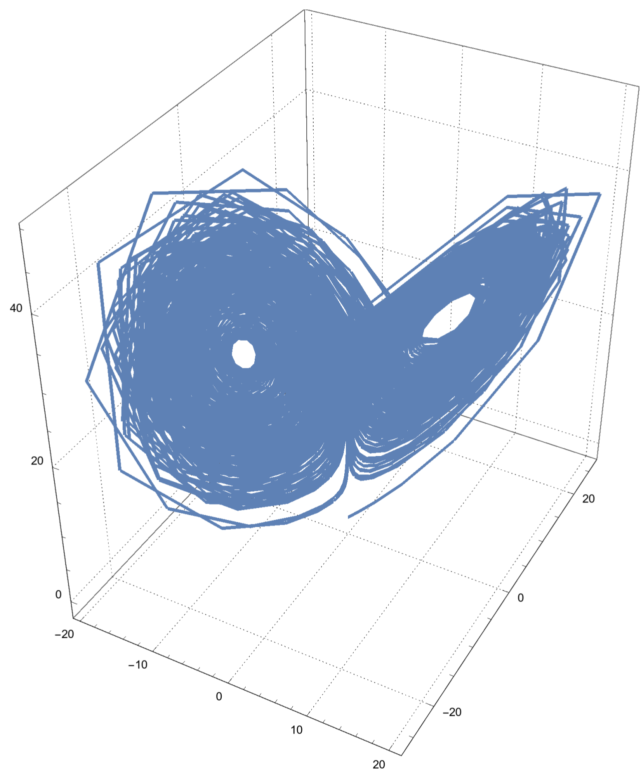

Figure 1: Trajectory plot around origin

2 Rotating hoop slider

Given the equation

$$mR^2\ddot{\alpha} + mR^2\left(\frac{g}{R} - \Omega^2\cos\alpha\right)\sin\alpha = 0 \tag{7}$$

we find a simplified Lyapunov candidate function as suggested by multiplying with $\dot{\alpha}$ and integrating in time:

$$\ddot{\alpha}\,\dot{\alpha} + \frac{g}{R}\sin(\alpha)\;\dot{\alpha} - \frac{\Omega^2}{2}\sin(2\alpha)\;\dot{\alpha} = 0$$
$$\frac{1}{2}\dot{\alpha}^2 - \frac{g}{R}\cos\alpha + \frac{\Omega^2}{4}\cos(2\,\alpha) + C = 0$$

In state space eq. (7) and this candidate Lyapunov function translate to:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \mathbf{f}(\mathbf{x}) = \begin{cases} x_2 \\ -\frac{g}{R}\sin x_1 + \frac{\Omega^2}{2}\sin(2x_1)\end{cases} \qquad \text{with } \mathbf{x} = \begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}\alpha\\ \dot{\alpha}\end{pmatrix} \tag{8}$$

$$V(\mathbf{x}) = \frac{1}{2}x_2^2 - \frac{g}{R}\cos x_1 + \frac{\Omega^2}{4}\cos(2\,x_1) + C \tag{9}$$

We now study the nonlinear stability of the previously found fixed point $\mathbf{x}^\star = \begin{pmatrix}0\\0\end{pmatrix}$, which turned out to be asymptotically stable in linear approximation for $\nu = \frac{R\,\Omega^2}{g} < 1$. The constant C in eq. (9) can always be chosen in such a way to guarantee $V(\mathbf{x}^\star) = 0$, in particular $C = \frac{g}{R} - \frac{\Omega^2}{4}$.

The second requirement for the function $V(\mathbf{x})$ to be a Lyapunov function is satisfied with the following condition:

$$\dot{V}(\mathbf{x}(t)) = \langle \mathrm{D}\,V(\mathbf{x})\,,\mathbf{f}(\mathbf{x})\rangle =$$
$$= \left\langle \left(\tfrac{g}{R}\sin x_1 - \tfrac{\Omega^2}{2}\sin(2x_1) \quad x_2\right), \left(x_2 \quad -\tfrac{g}{R}\sin x_1 + \tfrac{\Omega^2}{2}\sin(2x_1)\right)\right\rangle = 0 \le 0 \tag{10}$$
$$\tag{11}$$

Furthermore we need to have a local minimum in $\mathbf{x}^\star$, which can be validated e.g. by checking the definitness of the Hessian:

$$\mathrm{D}^2\,V(\mathbf{x})\big|_{\mathbf{x}=\mathbf{x}^\star} = \begin{bmatrix}\frac{g}{R}\cos x_1 - \Omega^2\cos(2x_1) & 0\\ 0 & 1\end{bmatrix}\Bigg|_{\mathbf{x}=\begin{pmatrix}0 & 0\end{pmatrix}} \ge 0$$
$$\iff \frac{g}{R} - \Omega^2 > 0 \iff \nu = \frac{R\,\Omega^2}{g} < 1$$

With the Hessian being positive definite in $\mathbf{x}^\star$ the fixed point is a local minimum for the Lyapunov function. Putting all these conditions together we can simply apply the first Lyapunov theorem (its assumptions are satisfied), which yields nonlinear stability for the fixed point $\mathbf{x}^\star$.

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2.1 Non asymptotic stability

If we further analyze why eq. (10) can just guarantee for $\dot{V}$ to be *semi* negative definite, we may analyze:

$$S = \{\mathbf{x} \in U_{\mathbf{x}^\star} \mid \dot{V}(\mathbf{x}) = 0\}$$

According to Krasowkij, if this set contained only the trivial trajectory $\mathbf{x}^\star$ itself, we could conclude asymptotic stability as well. But indeed, as eq. (10) shows, $\dot{V}$ remains zero for *any* trajectory:

$$\dot{V}(\mathbf{x}(t)) = \langle \mathrm{D}\, V(\mathbf{x}), \mathbf{f}(\mathbf{x}) \rangle = 0 \qquad \forall\, \mathbf{x} \in \mathbb{S} \times \mathbb{R} \tag{12}$$

Considering thus the surfaces on which $V(\mathbf{x}) = c$, with $c \in \mathbb{R}$ being some arbitrary constant value:

$$S_c = \{\mathbf{x} \in \mathbb{S} \times \mathbb{R} \mid V(\mathbf{x}) = c\} \tag{13}$$

The tangential directions $\hat{\mathbf{t}}(\mathbf{x})$ on these surfaces S_c must satisfy:

$$\langle \mathrm{D}\, V(\mathbf{x}), \hat{\mathbf{t}}(\mathbf{x}) \rangle = 0 \qquad \forall\, \mathbf{x} \in S_c$$

So comparing this result with eq. (12), we can state that the directions in which the solutions are traveling, actually are always exactly tangential to S_c. So even without knowing the flow map explicitly we can guarantee that:

$$F_{t_0}^{t}(S_c) = S_c \qquad \forall\, t \in \mathbb{R} \tag{14}$$

This means that any S_c as defined in eq. (13), is an *invariant* set of the system. In other words, even initial conditions arbitrarily close to the fixed point $\mathbf{x}^\star$ will *never* approach the fixed point, because they stay on their invariant set S_c. ◻ $q.e.d.$

3 Damped pendulum

Consider the damped pendulum equation:

$$\ddot{x} + c\,\dot{x} + \sin x = 0 \tag{15}$$

Changing to state space again, we have:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \mathbf{f}(\mathbf{x}) = \begin{cases} x_2 \\ -c\,x_2 - \sin x_1 \end{cases} \qquad \text{with } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \tag{16}$$

The total energy of the system can be expressed as the sum of potential an kinetic energy:

$$E_{tot}(\mathbf{x}) = \frac{1}{2}\, m\, x_2^2 + (1 - \cos x_1)\, l\, m\, g \tag{17}$$

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For simplicity, and to match with eq. (15), we assume all coefficients to be 1, i.e. $m = 1$ and $l = \frac{1}{g}$. We then pick $V(\mathbf{x}) = E_{tot}(\mathbf{x})$, i.e.:

$$V(\mathbf{x}) = \frac{1}{2} x_2^2 + 1 - \cos x_1 \tag{18}$$

We now check the conditions for being a Lyapunov function as before, considering the fixed point $\mathbf{x}^\star = \begin{pmatrix} 0 & 0 \end{pmatrix}$:

$$\begin{aligned} \dot{V}(\mathbf{x}(t)) &= \langle \mathrm{D}\, V(\mathbf{x}), \mathbf{f}(\mathbf{x}) \rangle = \\ &= \langle \begin{pmatrix} \sin x_1 & x_2 \end{pmatrix}, \begin{pmatrix} x_2 & -c\, x_2 - \sin x_1 \end{pmatrix} \rangle = -c\, x_2 < 0 \iff x_2 \neq 0 \end{aligned} \tag{19}$$

$$\mathrm{D}^2\, V(\mathbf{x})\big|_{\mathbf{x}=\mathbf{x}^\star} = \begin{bmatrix} \cos x_1 & 0 \\ 0 & 1 \end{bmatrix}\Bigg|_{\mathbf{x}=\begin{pmatrix} 0 & 0 \end{pmatrix}} > 0 \tag{20}$$

The Hessian being positive definite allows us again to conclude that $\mathbf{x}^\star$ *is* a local minimum for the chosen Lyapunov function, and therefor a neighborhood around the fixed points in which $V(\mathbf{x})$ is positive definite for sure exists.

However, eq. (19) no longer yields a strictly negative time derivative in a *neighborhood* $U_{\mathbf{x}^\star} \setminus \{\mathbf{x}^\star\}$, because in the two dimensional phase space we may leave the fixed point $\mathbf{x}^\star$ while keeping $x_2 = 0$. This means that we *cannot* conclude *asymptotic* stability using the Lyapunov's theorems.

3.1 Detailed reasoning

The energy of the *damped* pendulum is not conserved due to the damping term $c > 0$ in eq. (15). So we would expect the mechanical energy to decrease in time. However this damping term is proportional to velocity, which means that the mechanical energy decreases only for velocities $x_2 = \dot{x} \neq 0$. This does not allow us to construct a neighborhood in the phase space in which the derivative in time of the mechanical energy is strictly negative.

Nevertheless, compared with previous examples where energy was *added* to the system, and we had to modify the energy formulation in order to get at least a conservative system, for systems where energy is consumed, i.e. decreases in time, it still provides a reasonable Lyapunov candidate for showing stability. For *asymptotic* stability we may use of Krasovski's theorem.

3.2 Approach with Krasovski's theorem

To apply Krasovski's theorem with the previously chosen $V(\mathbf{x})$, we can use the obtained results for positive definiteness around $\mathbf{x}^\star$ and use $\dot{V} \leq 0$ for $x \in U$ in eq. (19). From there we know that $\dot{V} = -c\, x_2 = 0 \iff x_2 = 0$, which restricts the set $S = \left\{x \in U : \dot{V} = 0\right\}$ to the problematic neighborhood $\tilde{U}_x = \{x \in U : x_2 = 0,\ x_1 \neq 0\}$. Evaluating the state space equation eq. (16) in this region we get:

$$\dot{x}_2 = -c\, x_2 - \sin x_1 = -\sin x_1 \neq 0 \iff x_1 \neq 0 \tag{21}$$

This means that for *zero* velocity, in a position different from zero, trajectories immediately leave the set $S = \left\{x \in U : \dot{V} = 0\right\}$, due to the non zero derivative in x_2 direction shown in eq. (21). Physically this can be explained by the potential energy being non zero and converted into kinetic energy. So the set S indeed just consists of $\mathbf{x}^\star$. The theorem of Krasovski may therefor be applied and *asymptotic* stability can finally be concluded.

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4 Dirichlet's Theorem

We first recall the main quantities involved in the theorem's statement:

$$E(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\, \dot{\mathbf{q}}^\top \mathrm{M}(\mathbf{q})\, \dot{\mathbf{q}} + V(\mathbf{q}) = T(\mathbf{q}, \dot{\mathbf{q}}) + V(\mathbf{q})$$
$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\, \dot{\mathbf{q}}^\top \mathrm{M}(\mathbf{q})\, \dot{\mathbf{q}} - V(\mathbf{q}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q})$$

The equilibrium equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = 0 \tag{22}$$

which can be rewritten in terms of kinetic and potential energy as:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{\partial (T - V)}{\partial \mathbf{q}} = 0 \tag{23}$$

$$\frac{d}{dt}\frac{\partial \left(\frac{1}{2}\, \dot{\mathbf{q}}^\top \mathrm{M}(\mathbf{q})\, \dot{\mathbf{q}}\right)}{\partial \dot{\mathbf{q}}} - \frac{\partial (\frac{1}{2}\, \dot{\mathbf{q}}^\top \mathrm{M}(\mathbf{q})\, \dot{\mathbf{q}} - V)}{\partial \mathbf{q}} = 0 \tag{24}$$

4.1 State space for equilibrium equation

Explicitly writing the quadratic form of the matrix M, i.e. the kinetic energy, as a sum, we get:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{q}}}\left(\frac{1}{2}\, \dot{\mathbf{q}}^\top \mathrm{M}(\mathbf{q})\, \dot{\mathbf{q}}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{q}}}\left(\sum_{i,j}^{n} \frac{1}{2}\mathrm{M}_{ij}(\mathbf{q})\, \dot{\mathrm{q}}_i\, \dot{\mathrm{q}}_j\right)$$

Considering now the k-th row of this expression we get:

$$\begin{aligned}
\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathrm{q}}_k} = \frac{d}{dt}\frac{\partial}{\partial \dot{\mathrm{q}}_k}\left(\sum_{i,j}^{n} \frac{1}{2}\mathrm{M}_{ij}(\mathbf{q})\, \dot{\mathrm{q}}_i\, \dot{\mathrm{q}}_j\right) &= \frac{d}{dt}\sum_{j}^{n} \mathrm{M}_{kj}(\mathbf{q})\, \dot{\mathrm{q}}_j = \\
&= \sum_{j}^{n} \mathrm{M}_{kj}(\mathbf{q})\, \ddot{\mathrm{q}}_j + \sum_{j}^{n} \frac{d\,\mathrm{M}_{kj}(\mathbf{q})}{dt}\, \dot{\mathrm{q}}_j = \\
&= \sum_{j}^{n} \mathrm{M}_{kj}(\mathbf{q})\, \ddot{\mathrm{q}}_j + \sum_{i,j}^{n} \frac{\partial\, \mathrm{M}_{kj}}{\partial\, \mathrm{q}_i}(\mathbf{q})\, \dot{\mathrm{q}}_i\, \dot{\mathrm{q}}_j
\end{aligned}$$

given that in the first line:

- for $k \neq i$ and $k \neq j$ the derivatives are zero
- for $k = i = j$ the derivative can be simplified because we have a square form
- for $k = i \neq j$ the derivative can be simplified because we have it two times (given that M is symmetric)

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and simply applying the product derivation rule. Similarly we calculate the variation of the kinetic energy for a changing $\mathbf{q}$, in the k-the row we have:

$$\frac{\partial T}{\partial \mathrm{q}_k} = \frac{\partial}{\partial \mathrm{q}_k}\left(\sum_{i,j}^{n}\frac{1}{2}\mathrm{M}_{ij}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_j\right) = \sum_{i,j}^{n}\frac{1}{2}\,\frac{\mathrm{M}_{ij}}{\partial\,\mathrm{q}_k}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_j$$

since M is the only term depending on $\mathbf{q}$.

Equation (23) may therefore be rewritten as (considering its k-th row again):

$$\sum_{j}^{n}\mathrm{M}_{kj}(\mathbf{q})\,\ddot{\mathrm{q}}_j + \sum_{i,j}^{n}\left(\frac{\partial\,\mathrm{M}_{kj}}{\partial\,\mathrm{q}_i}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_j - \frac{1}{2}\,\frac{\mathrm{M}_{ij}}{\partial\,\mathrm{q}_k}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_j\right) + \frac{\partial\,V}{\partial\,\mathrm{q}_k} = 0 \tag{25}$$

So this turns out to be a second order ODE in time, which may be rewritten as a first order system, given that M is invertible by being positive definite. The respective states are $2n$ dimensional described by the extended vector $\begin{pmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{pmatrix}$.

4.2 Lyapunov candidate function

We want to show that if $V(\mathbf{q})$ admits a strict local minimum in a point $\mathbf{q_0}$, $\mathbf{q_0}$ is a (nonlinearly) stable equilibrium for the mechanical system.

For this purpose, we propose the following Lyapunov candidate function for eq. (23), just proofed to be eligible for state space with $\mathbf{q}$ and $\dot{\mathbf{q}}$:

$$E(\mathbf{q},\dot{\mathbf{q}}) = T(\mathbf{q},\,\dot{\mathbf{q}}) + V(\mathbf{q}) - V(\mathbf{q}_0) \tag{26}$$

Defining the Lyapunov function in this way, guarantees, with M being a positive definite matrix, such that $T(\mathbf{q},\,\dot{\mathbf{q}}) \geq 0$, the following conditions:

$$\begin{aligned} E(\mathbf{q},\dot{\mathbf{q}}) \geq 0 \qquad &\iff V(\mathbf{q}) - V(\mathbf{q}_0) \geq 0 \quad \forall\,\mathbf{q} \in U_{\mathbf{q}_0}; \quad \mathbf{q_0}\text{ local minimum} \\ E(\mathbf{q},\dot{\mathbf{q}})\big|_{(\mathbf{q}_0,0)} = 0 \qquad &\iff V(\mathbf{q}_0) - V(\mathbf{q}_0) = 0 \end{aligned}$$

Note that the kinetic energy T is zero, for 0 general velocity. The last criterium to fulfill is $\dot{E} \leq 0$. First, we calculate the time derivative of the kinetic energy T. By applying the chain rule we get:

$$\begin{aligned} \frac{d\,T\,(\mathbf{q}(t),\,\dot{\mathbf{q}}(t))}{dt} &= \sum_{k}^{n}\frac{\partial T}{\partial \mathrm{q}_k}\,\dot{\mathrm{q}}_k + \sum_{k}^{n}\frac{\partial T}{\partial \dot{\mathrm{q}}_k}\,\ddot{\mathrm{q}}_k \\ &= \sum_{k,i,j}^{n}\frac{1}{2}\,\frac{\mathrm{M}_{ij}}{\partial\,\mathrm{q}_k}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_j\dot{\mathrm{q}}_k + \sum_{k,j}^{n}\mathrm{M}_{kj}(\mathbf{q})\,\dot{\mathrm{q}}_j\ddot{\mathrm{q}}_k \end{aligned} \tag{27}$$

where previous results have been substituted. Due to symmetry we easily may exchange the indices of the matrix M which is multiplied by the $\ddot{\mathrm{q}}_k$ term. This allows us to use eq. (25) to substitute $\ddot{\mathrm{q}}_k$. Note that k and j needs to be exchanged for this, let us rewrite eq. (25) in a convenient manner, considering the j-th row now:

$$\sum_{k}^{n}\mathrm{M}_{jk}(\mathbf{q})\,\ddot{\mathrm{q}}_k = \sum_{i,k}^{n}\left(\frac{1}{2}\,\frac{\mathrm{M}_{ik}}{\partial\,\mathrm{q}_j}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_k - \frac{\partial\,\mathrm{M}_{jk}}{\partial\,\mathrm{q}_i}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_k\right) - \frac{\partial\,V}{\partial\,\mathrm{q}_j} \tag{28}$$

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Properly splitting now the last sum of eq. (27) we have:

$$\frac{d\,T(\mathbf{q}(t),\,\dot{\mathbf{q}}(t))}{dt} = \sum_{k,i,j}^{n} \frac{1}{2}\,\frac{\mathrm{M}_{ij}}{\partial\,\mathrm{q}_k}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_j\dot{\mathrm{q}}_k + \sum_{j}^{n}\left(\dot{\mathrm{q}}_j\,\sum_{k}^{n}\mathrm{M}_{kj}(\mathbf{q})\,\ddot{\mathrm{q}}_k\right)$$

which can now be used to properly substitute eq. (28):

$$\begin{aligned}\frac{d\,T(\mathbf{q}(t),\,\dot{\mathbf{q}}(t))}{dt} &= \sum_{k,i,j}^{n} \frac{1}{2}\,\frac{\mathrm{M}_{ij}}{\partial\,\mathrm{q}_k}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_j\dot{\mathrm{q}}_k + \sum_{j}^{n}\left(\dot{\mathrm{q}}_j\,\left(\sum_{i,k}^{n}\left(\frac{1}{2}\,\frac{\mathrm{M}_{ik}}{\partial\,\mathrm{q}_j}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_k - \frac{\partial\,\mathrm{M}_{jk}}{\partial\,\mathrm{q}_i}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_k\right) - \frac{\partial\,V}{\partial\,\mathrm{q}_j}\right)\right)\\ &= \sum_{k,i,j}^{n} \frac{1}{2}\,\frac{\mathrm{M}_{ij}}{\partial\,\mathrm{q}_k}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_j\dot{\mathrm{q}}_k + \sum_{i,j,k}^{n}\frac{1}{2}\,\frac{\mathrm{M}_{ik}}{\partial\,\mathrm{q}_j}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_j\,\dot{\mathrm{q}}_k - \sum_{i,j,k}^{n}\frac{\partial\,\mathrm{M}_{jk}}{\partial\,\mathrm{q}_i}(\mathbf{q})\,\dot{\mathrm{q}}_i\,\dot{\mathrm{q}}_j\,\dot{\mathrm{q}}_k - \sum_{j}^{n}\frac{\partial\,V}{\partial\,\mathrm{q}_j}\,\dot{\mathrm{q}}_j\end{aligned}$$

Of course, summation indices may be arbitrarily exchanged, which is why the first three terms together sum up to 0. It follows that:

$$\frac{d\,T(\mathbf{q}(t),\,\dot{\mathbf{q}}(t))}{dt} = -\sum_{j}^{n}\frac{\partial\,V}{\partial\,\mathrm{q}_j}\,\dot{\mathrm{q}}_j \tag{29}$$

Now for the potential energy we have:

$$\frac{d\,V(\mathbf{q}(t))}{dt} = \sum_{j}^{n}\frac{\partial\,V}{\partial\,\mathrm{q}_j}\,\dot{\mathrm{q}}_j \tag{30}$$

Combining the eqs. (29) and (30) we finally get:

$$\frac{d\,E(\mathbf{q},\,\dot{\mathbf{q}})}{dt} = \frac{d\,T(\mathbf{q},\,\dot{\mathbf{q}})}{dt} + \frac{d\,V(\mathbf{q})}{dt} = -\sum_{j}^{n}\frac{\partial\,V}{\partial\,\mathrm{q}_j}\,\dot{\mathrm{q}}_j + \sum_{j}^{n}\frac{\partial\,V}{\partial\,\mathrm{q}_j}\,\dot{\mathrm{q}}_j = 0 \le 0 \tag{31}$$

So we finally got that energy is *conserved* in the given system.

4.3 Conclusion

Recapitulating, we have proofed that

$$E(\mathbf{q},\dot{\mathbf{q}}) = T(\mathbf{q},\,\dot{\mathbf{q}}) + V(\mathbf{q}) - V(\mathbf{q}_0) \tag{32}$$

is a valid Lyapunov function for the system, because:

(i) $E(\mathbf{q},\dot{\mathbf{q}}) \ge 0$, in a neighborhood of $(\mathbf{q}_0,\mathbf{0})$ using the fact that M is positive definite and $V(\mathbf{q})$ admits a local minimum $\mathbf{q}_0$

(ii) $E(\mathbf{q}_0,\mathbf{0}) = 0$

(iii) $\frac{d\,E(\mathbf{q},\,\dot{\mathbf{q}})}{dt} = 0 \le 0$, valid actually for *any* admissible $\mathbf{q}$; during the derivation it has been exploited that M is symmetric

This allows us to apply Lyapunov's Theorem I to conclude that $\mathbf{q}_0$ is a stable fixed point of the nonlinear system.

□ *q.e.d.*

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