ETH zürich IMES

Nonlinear Dynamics and Chaos I. Assignment 2

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1 Approximation of forced oscillator

Considering the equation

$$\begin{cases} \ddot{x} + \omega_0^2\, x = \varepsilon\, M\, x^2 \\ \dot{x}(0) = 0 \\ x(0) = a_0 \end{cases} \tag{1}$$

we observe that the system equations will be found to be $\mathcal{C}^\infty$ in ε, which is why we may generally approximate the solution in orders of ε. As suggested we use Lindstedt's Ansatz and introduce a new time as well:

$$\begin{aligned} x_\varepsilon(t) &= \varphi_0(t) + \varepsilon\varphi_1(t) + \mathcal{O}(\varepsilon^2) \\ \varphi_i(t) &= \varphi_i(t + T_\varepsilon) \\ T_\varepsilon &= \frac{2\pi}{\omega_\varepsilon} \\ \tau &= \omega_\varepsilon t \\ \Rightarrow \frac{d}{dt} &= \omega_\varepsilon \frac{d}{d\tau} \end{aligned}$$

Simultaneously we approximate ω_ε as:

$$\omega_\varepsilon = \omega_0 + \varepsilon\, \omega_1 + \mathcal{O}(\varepsilon^2)$$

where ω_0 is the natural frequency of the equation given in eq. (1).

By substituting into eq. (1) all the introduced approximations, denoting the derivative in τ with $'$ and neglecting all higher order terms, we get:

$$(\omega_0^2 + 2\,\varepsilon\,\omega_0\,\omega_1)\varphi'' + \varepsilon\,\omega^2\varphi_1'' + \omega^2\,\varphi_0 + \varepsilon\,\omega^2\,\varphi_1 = \varepsilon\,M\,\varphi_0^2 \tag{2}$$

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1.1 Order 0 approximation

For the order 0 we simply get a homogeneous oscillator with natural frequency equals to 1 (in the new time, i.e. $\varphi(\tau)$:

$$\begin{cases} \omega_0^2\,\varphi_0'' + \omega_0^2\varphi_0 = 0 \quad \Rightarrow \quad \varphi_0'' + \varphi_0 = 0 \\ \dot{\varphi}_0(0) = 0 \\ \varphi_0(0) = a_0 \end{cases} \tag{3}$$

The solution to this homogeneous differential equation satisfying the initial conditions is:

$$\varphi_0 = a_0 \cos\tau \tag{4}$$

1.2 Order ε approximation

Substituting the solution (eq. (4)) back into eq. (2), we get (rearranging):

$$\begin{cases} \varphi_1'' + \varphi_1 = \frac{M\,a_0^2}{\omega_0^2}\cos^2\tau + 2\frac{\omega_1}{\omega_0}a_0\cos\tau \\ \varphi_1'(0) = 0 \\ \varphi_1(0) = 0 \end{cases} \tag{5}$$

Note that we the initial conditions are already satisfied by φ_0, which is why they need to be zero here. In this way the solution of the homogeneous equation simply collapses. Furthermore we can reduce the first trigonometric term:

$$\varphi_1'' + \varphi_1 = \frac{M\,a_0^2}{2\,\omega_0^2} + \frac{M\,a_0^2}{\omega_0^2}\cos(2\,\tau) + 2\frac{\omega_1}{\omega_0}a_0\cos\tau \tag{6}$$

We now notice that the eventual solution for the last term would violate our Ansatz: we observe resonance, since $t\,\cos\tau$ would need to be used. To avoid this we pick

$$\omega_1 = 0 \tag{7}$$

We select our particular solution to be in the form of:

$$\varphi_{1p} = A + B\cos(2\,\tau) \tag{8}$$

Coefficients can easily be determined to be:

$$A = \frac{M\,a_0^2}{2\,\omega_0^2}$$
$$B = -\frac{M\,a_0^2}{3\omega_0^2}$$

The $\mathcal{O}(\varepsilon)$ solution is therefor (note that with $\omega_1 = 0$ we simply have $\tau = \omega_0\,t$):

$$\varphi(t) = a_0\cos(\omega_0\,t) + \varepsilon\frac{M\,a_0^2}{2\,\omega^2} - \frac{M\,a_0^2}{3\omega_0^2}\cos(2\,\omega_0\,t) \tag{9}$$

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2 Van der Pol equation

Given the van der Pol equation

$$\begin{cases} \ddot{x} + \varepsilon\,(x^2 - 1)\,\dot{x} + x = F\cos(\omega\, t) \\ \dot{x}(0) = 0 \\ x(0) = a_0 \end{cases} \tag{10}$$

as before we notice that the system equations should be differentiable in ε, which is why we might approximate

$$x(t) = \varphi_0(t) + \varepsilon\varphi_1(t) + \mathcal{O}(\varepsilon)^2 \tag{11}$$

Substituting this in eq. (10)

$$(\varphi_0'' + \varphi_0) + \varepsilon\left(\left({\varphi_0}^2 - 1\right)\varphi_0' + \varphi_1'' + \varphi_1\right) + \mathcal{O}\left(\varepsilon^2\right) = F\cos(\omega\, t) \tag{12}$$

Note that in this case we did not change the time, and $'$ just denotes the common time derivative.

2.1 Order 0 approximation

For the $\mathcal{O}(0)$ approximation eq. (12) collapses to:

$$\varphi_0''(t) + \varphi_0(t) = F\cos(t\omega) \tag{13}$$

This ODE, with the initial value conditions has a general solution:

$$a_0\cos(t) + \frac{F(\cos(t) - \cos(t\omega))}{\omega^2 - 1} \tag{14}$$

Now using the suggestions, we avoid $2\,\pi$ - periodic terms and aim for an exact $\frac{2\pi}{\omega}$ - periodic solution. For this reason we may select a_0 to be:

$$a_0 = -\frac{F}{\omega^2 - 1} \tag{15}$$

In this way our constant order approximation in ε becomes:

$$\varphi_0(t) = \frac{F}{\omega^2 - 1}\cos(\omega\, t) \tag{16}$$

2.2 Order ε approximation

For the next order we simply substitute the obtained solution in the original eq. (12). This yields after rearranging terms:

$$\frac{F\omega\sin(t\omega)\left(F^2\cos^2(t\omega) - (\omega^2 - 1)^2\right)}{\left(\omega^2 - 1\right)^3} + \varphi_1''(t) + \varphi_1(t) = 0 \tag{17}$$

The homogeneous solutions would introduce again some non $\frac{2\pi}{\omega}$ - periodic component, i.e. $\cos(t)$ and $\sin(t)$ so we simply choose them to be zero.

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As we can see the equation has become a bit more complicated, with a cosine squared multiplied by a sinus, which again need to be reduced. After some notable amount of calculations, the particular solution can be found to be:

$$\varphi_1(t) = \frac{F\omega \sin(\omega t)\left(F^2(\omega^2-1)\cos(2\omega t) + F^2(5\omega^2-1) - 2(\omega^2-1)^2(9\omega^2-1)\right)}{2(\omega^2-1)^4(9\omega^2-1)} \tag{18}$$

So finally putting the solutions together by adding $\varepsilon\varphi_1$ to the constant order solution, we get:

$$x(t) \approx \frac{F\cos(\omega t)}{1-\omega^2} + \frac{\varepsilon F \omega \sin(\omega t)\left(F^2(\omega^2-1)\cos(2\omega t) + F^2(5\omega^2-1) - 2(\omega^2-1)^2(9\omega^2-1)\right)}{2(\omega^2-1)^4(9\omega^2-1)} \tag{19}$$

2.3 Solution comparison plot

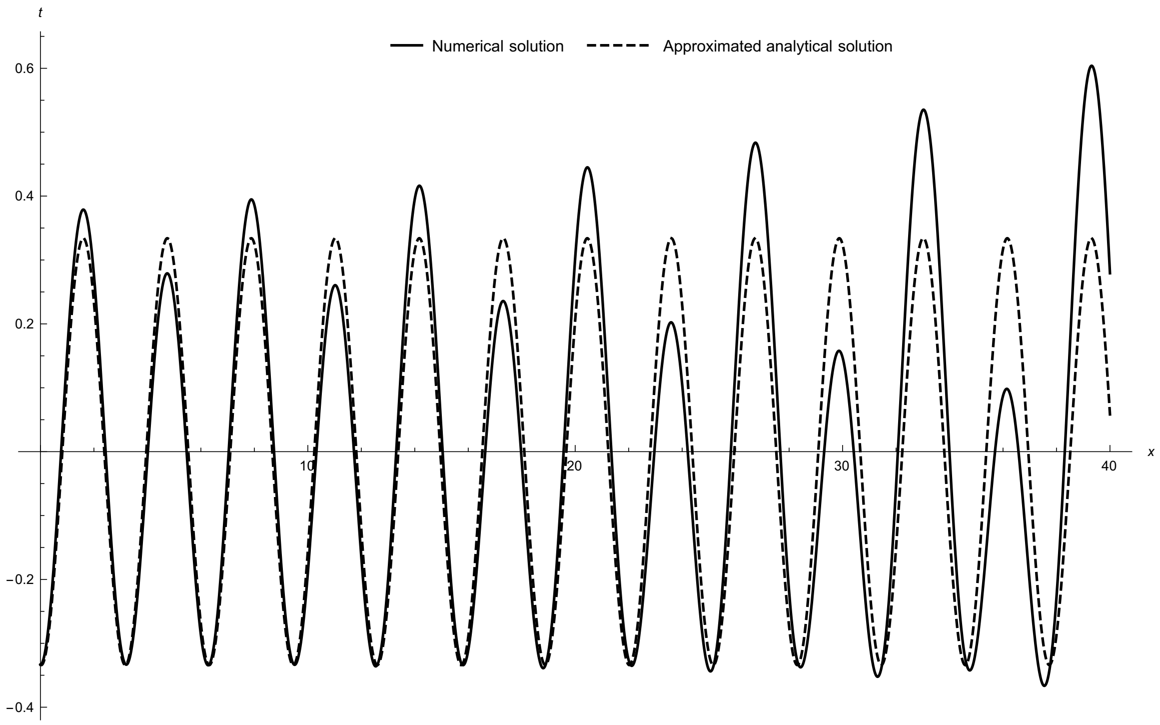

Figure 1: Solution Comparison

As fig. 1 suggests, eq. (19) can be a reasonable approximation in regions around t=0, but for the $\mathcal{O}(\varepsilon)$ approximation, the (actual) numerical solution diverges in a non trivial behavior from this perfectly periodic approximation.

3 Mass on a loop equilibria analysis

Converting the equation

$$mR^2\ddot{\alpha} + bR^2\dot{\alpha} + mR^2\left(\frac{g}{R} - \Omega^2\cos\alpha\right)\sin\alpha = 0 \tag{20}$$

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into a first order system of ODEs, eq. (20) becomes:

$$\dot{x} = f(x) \qquad x = \begin{bmatrix} \alpha \\ \dot{\alpha} \end{bmatrix} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \sin x_1 \left(\Omega^2 \cos x_1 - \frac{g}{R} \right) - \frac{b x_2}{m} \end{cases}$$

Looking for equilibria points where

$$f(x^\star) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\iff$

$$x_2^\star = \dot{\alpha}^\star = 0 \tag{21}$$

$$x_1^\star = \alpha^\star = \begin{cases} 2k\pi & k \in \mathbb{Z} \\ \arccos \frac{1}{\nu} \end{cases} \tag{22}$$

with

$$\nu = \frac{R\,\Omega^2}{g} \tag{23}$$

Plotting the position α in function of ν yields:

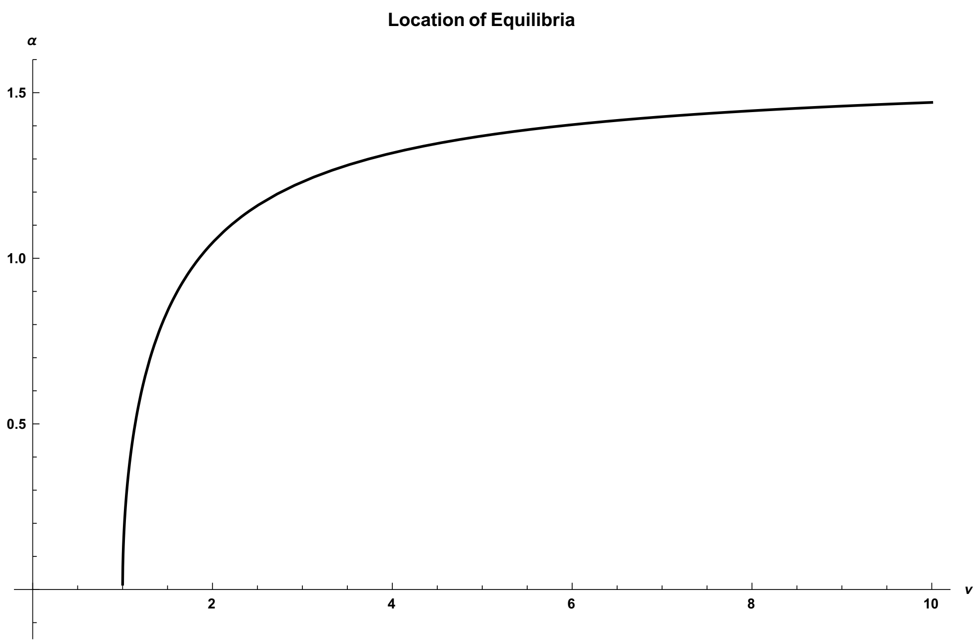

Figure 2: Location of equilibria

As we can see, also analytically, values of ν need to be ≥ 1 in order to meet the domain requirements of arccos. The angular postition α is approaching $\frac{\pi}{2}$, while demanding Ω^2 speed or large R, as eq. (23) shows.

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3.1 Stability analysis

The general Jacobian of section 3 is:

$$J(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ \Omega^2 \cos(2x_1) - \frac{g}{R}\cos x_1 & -\frac{b}{m} \end{bmatrix} \tag{24}$$

Plugging in the equilibria point $(\arccos(\frac{1}{\nu}), 0)$ we get:

$$J(x_1^\star, x_2^\star) = \begin{bmatrix} 0 & 1 \\ \frac{g^2}{R^2\Omega^2} - \Omega^2 & -\frac{b}{m} \end{bmatrix} \tag{25}$$

Getting the associated characterstic equation of the eigenvalues λ:

$$\begin{aligned} \det\left(J(x_1^\star, x_2^\star) - \lambda\mathbb{I}\right) &= 0 \\ \begin{vmatrix} -\lambda & 1 \\ \frac{g^2}{R^2\Omega^2} - \Omega^2 & -\frac{b}{m} - \lambda \end{vmatrix} &= 0 \\ \lambda^2 + \frac{b}{m}\lambda + \left(\Omega^2 - \frac{g^2}{R^2\Omega^2}\right) &= 0 \end{aligned}$$

Applying Routh-Hurwitz, we have:

$$\begin{aligned} D_2 &= \begin{vmatrix} \frac{b}{m} & \left(\Omega^2 - \frac{g^2}{R^2\Omega^2}\right) \\ 0 & 1 \end{vmatrix} = \frac{b}{m} > 0 \\ D_1 &= \frac{b}{m} > 0 \\ D_0 &= \left(\Omega^2 - \frac{g^2}{R^2\Omega^2}\right) > 0 \end{aligned}$$

From D_0 we get:

$$\begin{aligned} \Omega^2 &> \frac{g^2}{R^2\Omega^2} \\ \frac{\Omega^2 R}{g} &> \frac{g}{\Omega^2 R} \\ \nu &> \frac{1}{\nu} \iff \nu > 1 \end{aligned}$$

From the other equilibria point found in eq. (20), namely $(2k\pi, 0)$, we get from the jacobian in eq. (24):

$$J(x_1^\star, x_2^\star) = \begin{bmatrix} 0 & 1 \\ \Omega^2 - \frac{g}{R} & -\frac{b}{m} \end{bmatrix}$$

With a completely analog development to the one carried out before, we get for asymptotic stability the condition:

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$$\frac{g}{R} - \Omega^2 > 0 \tag{26}$$

$$\frac{g}{R} > \Omega^2 \tag{27}$$

$$1 > \frac{R\Omega^2}{g} \tag{28}$$

$$\nu < 1 \tag{29}$$

Summerizing the results we have:

- for $(x_1^\star, x_2^\star) = (\arccos(\frac{1}{\nu}), 0)$ asymptotically stable if $\nu > 1$
- for $(x_1^\star, x_2^\star) = (2k\pi, 0)$ asymptotically stable if $\nu < 1$

Note that given the parameters of the system $\nu > 0$ in any case. In $\nu = 1$ we have a biforcation for which we cannot determine the stability of the non-linear system in the discussed way, because for both fixpoints the associated matrix gets singular, meaning that they are no hyperbolic fixpoints anymore. Hartman-Grobman theorem for topological equivalence can thus not be applied anymore.

For our graph in fig. 2 this means that we can add a segment $0 < \nu < 1$ with $\alpha = 0$ (even $2k\pi$), since under these conditions it turns out to be an asymptotically stable fixpoint as well.

4 Discrete Dynamical Systems

In this section the autonomous system

$$x_{k+1} = f(x_k), \qquad f : \mathbb{R}^n \to \mathbb{R}^n, \qquad x \in \mathbb{R}^n \tag{30}$$

is treated.

4.1 Linearized mapping

To define a linearized mapping for the vicinity of p we pick:

$$y_k = x_k - p \tag{31}$$

We have:

$$\begin{aligned} y_{k+1} &= x_{k+1} - p \\ &= f(x_k) - f(p) \\ &= f(p + y_k) - f(p) \\ &= f(p) + D_x f(p) y_k - f(p) \\ &= D_x f(p) y_k \end{aligned}$$

Note that y_k can be considered to be small, since $y_k = x_k - p$. Defining $A := D_x f(p)$, we have

$$y_{k+1} = A\, y_k \tag{32}$$

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4.2 General solution for A fully ranked

Being s_i one of the n independent eigenvectors of A, by the definition of eigenvector / value we may write:

$$A\, s_i = \lambda_i\, s_i \tag{33}$$

Defining V as matrix with all n vectors s_i in the columns, i.e.

$$V := \begin{bmatrix} s_1 & s_2 & \dots & s_n \end{bmatrix}$$

we may apply the definition eq. (33) using the whole matrix V getting

$$\begin{aligned} A\, V &= \begin{bmatrix} \lambda_1 s_1 & \lambda_2 s_2 & \dots & \lambda_n s_n \end{bmatrix} \\ &= V \Lambda \end{aligned} \tag{34}$$

where Λ is a diagonal matrix with all the eigenvalues $\lambda_i \in \mathbb{C}$ on the diagonal.
Since as the hypothesis states, there are *n linearly independent* eigenvectors, V is fully ranked $\Rightarrow \exists V^{-1}$. Right multiplying eq. (34) by the inverse of V, yields

$$A = V\, \Lambda\, V^{-1} \tag{35}$$

Now starting from $k = 0$, using eq. (35) yields:

$$\begin{aligned} y_1 &= A\, y_0 \\ y_2 &= A\, y_1 = A\, A\, y_0 = \\ &= V\, \Lambda\, V^{-1}\, V\, \Lambda\, V^{-1}\, y_0 \\ &= V\, \Lambda^2\, V^{-1}\, y_0 \end{aligned}$$

Where Λ^2 is a diagonal matrix with all the eigenvalues *elevated to the power of 2* on its diagonal.
In general:

$$y_k = V\, \Lambda^k\, V^{-1}\, y_0 \tag{36}$$

Inspecting the product matrix product $V\, \Lambda^k$ we notice that the result is made of the following columns:

$$V\, \Lambda^k = \begin{bmatrix} \lambda_1^k s_1 & \lambda_2^k s_2 & \dots & \lambda_n^k s_n \end{bmatrix} \tag{37}$$

For sure the solution is a linear combination of those columns, i.e.

$$y_k = c_1 \lambda_1^k s_1 + c_2 \lambda_2^k s_2 + \cdots + c_n \lambda_n^k s_n \tag{38}$$

Indeed, the coordinates c_i can be defined as:

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = V^{-1}\, y_0 \tag{39}$$

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Note that we might want to redefine eq. (37) as:

$$\Psi(k) := V\,\Lambda^k \tag{40}$$

which then yields

$$y_k = \Psi(k)\,\Psi^{-1}(0)\,y_0 \tag{41}$$

being $\Psi^{-1}(0)$ nothing else but V^{-1}.

□ *q.e.d.*

4.3 Stability definitions

4.3.1 Stable

We call the fixed point $p = f(p)$ stable if

$$\begin{gathered} \forall\, k > 0, \quad \forall\, \varepsilon > 0 \quad \exists \quad \delta(\varepsilon) > 0 \\ \text{such that} \quad \forall\, x_0 \in \mathbb{R}^n \quad \text{with} \quad |x_0 - p| \le \delta \quad \text{we have} \\ |x_k - p| = |f^k(x_0) - p| \le \varepsilon \end{gathered}$$

4.3.2 Asymptotically stable

We call the fixed point $p = f(p)$ asymptotically stable if

1. it is stable
2. $$\begin{gathered} \forall\, k > 0, \quad \exists \quad \delta_0(k) > 0 \\ \text{such that} \quad \forall\, x_0 \in \mathbb{R}^n \quad \text{with} \quad |x_0 - p| \le \delta_0 \\ \lim_{k\to\infty} x_k = \lim_{k\to\infty} f^k(x_0) = p \end{gathered}$$

4.3.3 Unstable

A fixed point $p = f(p)$ is called unstable, if it is not stable.

4.4 Sufficient and necessary condition for asymptotic stability

The linearized system defined in eq. (32) with A fully ranked, i.e. all n eigenvectors being linearly independent, is asymptotically stable, $\iff$

$$|\lambda_i| < 1 \quad \forall \quad 1 \le i \le n, \quad \lambda_i \in \mathbb{C} \tag{42}$$

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4.4.1 Proof 1: Sufficient

The definition of y given in eq. (31) tells us that generally, our fixed point for the linearized system is found in $y = 0$. We need then to apply the definition of asymptotic stability to this system, i.e.[1]:

$$y_k = \lim_{k\to\infty} f^k(y_0) \overset{?}{=} 0$$

Where y_0 is a small perturbation $\leq \delta_0$ of the initial condition, i.e. near zero. From eq. (41) we get:

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} \Psi(k)\,\Psi^{-1}(0)\,y_0 \tag{43}$$

Using eq. (38) and the linear properties of the limit operator we get:

$$\lim_{k\to\infty} c_i \lambda_i^k s_i \overset{?}{=} 0 \quad 1 \leq i \leq n \tag{44}$$

The limit just exists finitely, if

$$|\lambda_i| < 1 \quad 1 \leq i \leq n \tag{45}$$

□ $q.e.d.$

4.4.2 Proof 2: Necessary

Equation (38), might suggest that there is another possibility of all c_i being zero. Recalling eq. (39), denoting all the coordinates by the vector c we get:

$$c = V^{-1}\,y_0 \tag{46}$$

Remember that V is fully ranked, being the matrix with the eigenvectors in its columns, and so is V^{-1}. Indeed the fixed point itself being zero, generates coordinates all being zero as well. However, the nullspace is empty elsewhere due to the full rank, and it is indeed just the trivial subspace of the zero vector (fixed point) itself. This means that any *smallest perturbation* in an arbitrary direction generates arbitrary values different from zero for any of the coordinates. This means that we will never be able to find a ball defined by $\delta_0 > 0$ in which we could guarantee that even just some coordinates would be zero.

This means that the condition reported in eq. (45) is indeed *necessary* to guarantee an asymptotically stable fixpoint.

□ $q.e.d.$

[1]question mark denotes assumption to proof

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