ETH zürich

IMES

Nonlinear Dynamics and Chaos I. Assignment 1

Florian Mahlknecht

2019-10-08

Contents

1 Gronwall's inequality

Being

$$x(t, x_0), \qquad x(t, \hat{x}_0)$$

two solutions of the differential equation

$$\dot{x} = f(x, t), \qquad x \in \mathbb{R}^n \tag{1}$$

they satisfy

$$\frac{dx}{dt}(t, x_0) = f(x(t, x_0), t)$$
$$\frac{dx}{dt}(t, \hat{x}_0) = f(x(t, \hat{x}_0), t)$$

By integrating the equations from t_0 to t, we get

$$\int_{t_0}^{t} \frac{dx}{dt'}(t', x_0)dt' = \int_{t_0}^{t} f(x(\tau, x_0), \tau)d\tau$$
$$\int_{t_0}^{t} \frac{dx}{dt'}(t', \hat{x}_0)dt' = \int_{t_0}^{t} f(x(\tau, \hat{x}_0), \tau)d\tau$$

According to the Fundamental Theorem of Calculus, we get

fmahlknecht@student.ethz.ch
19-945-351

$$x(t, x_0) - x(t_0, x_0) = \int_{t_0}^{t} f(x(\tau, x_0), \tau) d\tau$$
$$x(t, \hat{x}_0) - x(t_0, \hat{x}_0) = \int_{t_0}^{t} f(x(\tau, \hat{x}_0), \tau) d\tau$$

In t_0 by definition of the Cauchy problem, we get x_0 and $\hat{x}_0$ respectively. Considering now the normed difference of the two equations, to investigate how they diverge, we get (bringing x_0 and $\hat{x}_0$ to the right side):

$$\begin{aligned}
|x(t, x_0) - x(t, \hat{x}_0)| &= |x_0 - \hat{x}_0 + \int_{t_0}^{t} [f(x(\tau, x_0), \tau) - f(x(\tau, \hat{x}_0), \tau)] d\tau| \\
&\leq |x_0 - \hat{x}_0| + \int_{t_0}^{t} |f(x(\tau, x_0), \tau) - f(x(\tau, \hat{x}_0), \tau)| d\tau
\end{aligned}$$

Further, we want to use the given hypothesis and pick a Lipschitz constant $L > 0$ for the function f over the domain of trajectories in the time interval $[t_0, t]$, which guarantees:

$$|f(x(\tau, x_0), \tau) - f(x(\tau, \hat{x}_0), \tau)| \leq L|x(\tau, x_0) - x(\tau, \hat{x}_0)|$$

Using this property we get:

$$|x(t, x_0) - x(t, \hat{x}_0)| \leq |x_0 - \hat{x}_0| + \int_{t_0}^{t} L|x(\tau, x_0) - x(\tau, \hat{x}_0)| d\tau \tag{2}$$

Now going back to the Gronwall's inequality

$$u(t) \leq C + \int_{t_0}^{t} u(\tau)v(\tau) d\tau \quad \Rightarrow \quad u(t) \leq C\, e^{\int_{t_0}^{t} v(\tau) d\tau} \tag{3}$$

we satisfy the conditions in eq. (2) of $C \geq 0$ and $u(t)$, $v(t)$ being positive scalar functions, in particular with:

$$\begin{aligned}
C &= |x_0 - \hat{x}_0| \\
u(t) &= |x(t, x_0) - x(t, \hat{x}_0)| \\
v(t) &= L
\end{aligned}$$

So finally applying eq. (3) yields:

$$|x(t, x_0) - x(t, \hat{x}_0)| \leq |x_0 - \hat{x}_0|\, e^{\int_{t_0}^{t} L d\tau} = |x_0 - \hat{x}_0|\, e^{L(t - t_0)} \tag{4}$$

$\square$ *q.e.d.*

Florian Mahlknecht
fmahlknecht@student.ethz.ch
19-945-351

2 Pendulum Wall

The pendulum hitting against the wall, in fig. 1 from the assignment, has to be considered.

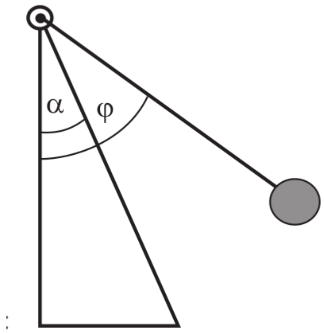


Figure 1: Pendulum Sketch

The later involved coefficient of restitution c_k[1] relates inbound and outbound velocities to each other, i.e.

$$c_k = \frac{v_{out}}{v_{in}} \tag{5}$$

Figure 4 sketches the phase space of the system for the first part with $c_k = 1$.

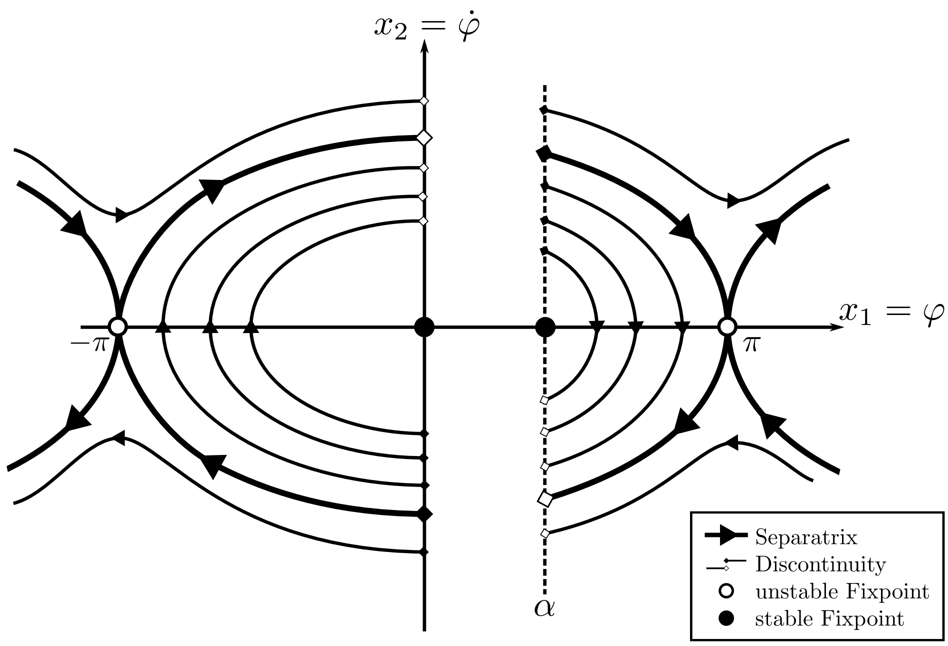


Figure 2: Phase Space Draft $c_k = 1$, $\alpha > 0$

To apply a mathematically correct meaning to the drafted phase space, two things are important to note:

[1]often denoted also as e

fmahlknecht@student.ethz.ch
19-945-351

- the bold drafted separatrix does never actually touch the unstable fixpoints, being themselves unique solutions to the problem, but they might come infinitesimally close (for simplicity this is neglected in the actual drawing)
- the given separatrix may not technically be recognized as a separatrix, because it contains discontinuities and therefor non differentiable points, making it not eligible for being a connected manifold. For better understanding, they are drawn nevertheless, given that they still separate different behaviors.

Using the hypothesis that solutions instantaneously jump to the opposite velocity values in $\varphi = \alpha$ and $\varphi = 0$, we can simply draw the parts of the already known ellipsis adding discontinuities in the contact positions. In this way solutions oscillating below the unstable fixpoint $\varphi = \pi$ simply continue for an infinite time on their trajectories, given that there is no friction. Note that in fig. 2, three distinct inner solutions are sketched, jumping always back in their own trajectory. In the interval between 0 and α, we do not have any solutions, given the mechanical obstacle shown in fig. 1. In a similar way solutions outside the separatrix jump back and forth between α and 0 (note the periodicity of the angular position space).

If we consider the same diagram for some $\alpha < 0$, given in fig. 3, we may notice the point symmetry of 180°around the origin. This feels quite natural, an indeed directions are preserved in this way.

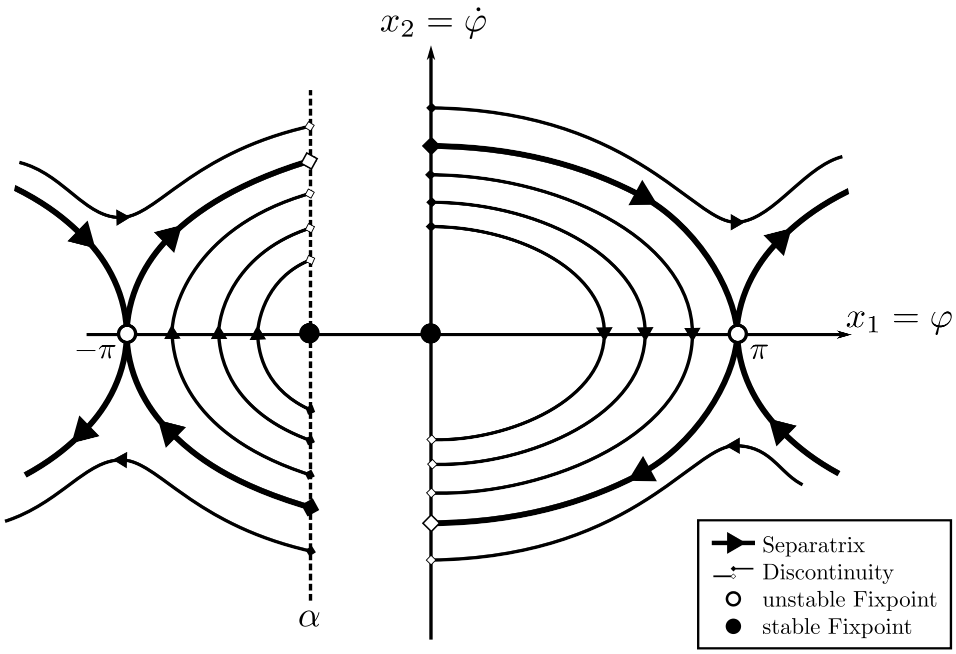


Figure 3: Phase Space Draft $c_k = 1$, $\alpha < 0$

Florian Mahlknecht
fmahlknecht@student.ethz.ch
19-945-351

Treating now the case in which the coefficient of restitution c_k becomes 0.5, we have quite a different phase space picture, see fig. 4.

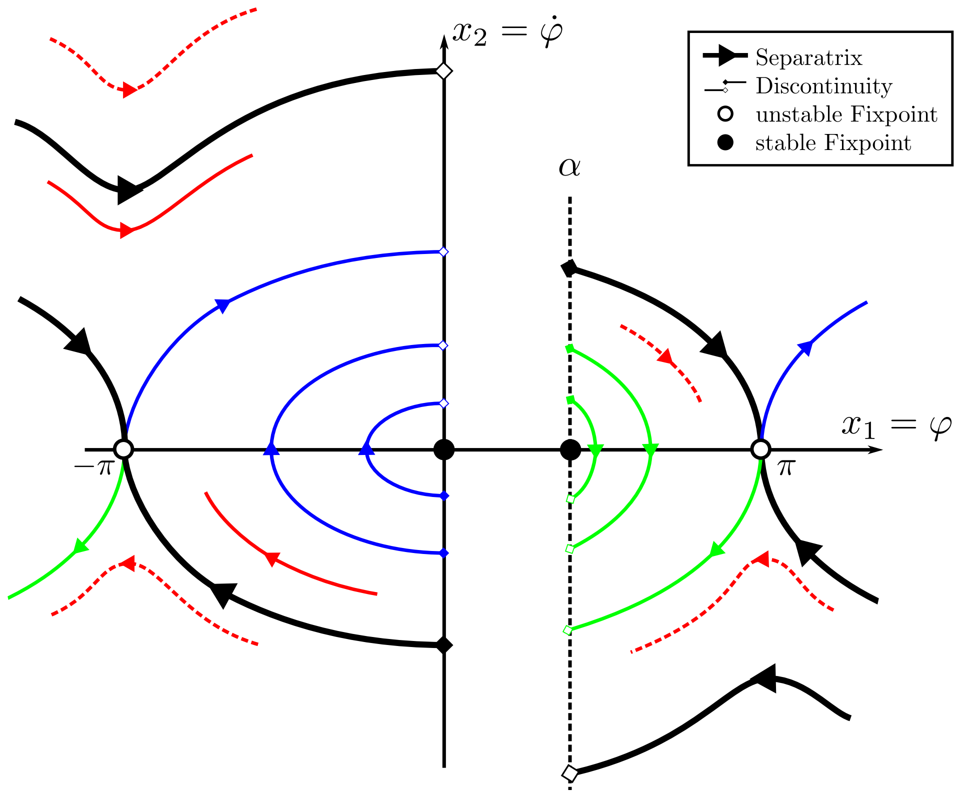


Figure 4: Phase Space Draft $c_k = 0.5$, $\alpha > 0$

As discussed before, the same assumtion for non touching solutions and the non-rigorous definition of separatrix hold. In this case indeed the separatrices become more complicated, but they still divide different behaviors.

First, on the inner side, below the critical ellipsis passing through π and $-\pi$ we have solutions which all approach either the stable fixpoint $\varphi = \alpha$ or $\varphi = 0$, depending on which side they start. Given that there is still no friction, they follow the same elliptical trajectory as before, until they hit and loose half of their speed in post-impact condition. In this way the blue colored solution hits the wall in $\varphi = 0$ and the red sketched solution hits the wall in $\varphi = \alpha$. Note that they can both start arbitrarily close from the critical points $\varphi = \pi$ or $-\pi$ respectively, *without touching them*. Furthermore they need to face the right direction, i.e. slightly with $\dot{\varphi} < 0$ in case of the green one and $\dot{\varphi} > 0$ in case of the blue one.

The most interesting part is the fact that the starting part of those solutions (green and blue) are no longer a separatrix: even with some $\dot{\varphi} << 0$ the green solution would still hit the wall in α. The crucial fact is that even the separatrices loose half of there velocity when hitting the walls. In this way there exists a whole labyrinth along the velocity space ($= \mathbb{R}$) of separatrices which basically decide if solutions are going to end up in the wall in $\varphi = 0$ or $\varphi = \alpha$. This is illustrated in the red solutions, distinct in one continuous and one dashed sketched line.

As a last fact, probably worth being mentioned, the velocity losses in the impacts on the walls are of an exponentially decaying nature. This means that the stable fix points in α and 0 are reached just after an infinite amount of time.

fmahlknecht@student.ethz.ch
19-945-351

3 Damped Pendulum

The exercise considers the following damped and enforced pendulum equation:

$$\ddot{\theta} + k\,\dot{\theta} + \sin\theta = a\sin t \tag{6}$$

with k and a being two positive scalar parameters.
In order to rewrite this equation as a system of first order ODEs, we may introduce:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$$

This allows to rewrite eq. (6) as follows:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) = \begin{cases} x_2 \\ -k\,x_2 - \sin x_1 + a\sin t \end{cases} \tag{7}$$

3.1 Implementation Details

For solving eq. (7) for different initial conditions with just one function call to `ode45` we may use an extended implementation of eq. (7) supporting long initial condition vectors.[2] Details are presented in listing 1. Note that the argument `par` is nothing but a vector containing $\begin{pmatrix} k & a \end{pmatrix}$.

```
function xdot = pendulumODE(t,x,par)
%PENDULUMODE Defines the ODE of the damped pendulum
%   defines it in such a way to handle also long initial condition vectors
[m,n] = size(x);
x = reshape(x, [2 m/2]);
xdot(1,:) = x(2,:);
xdot(2,:) = -par(1).*x(2,:) -sin(x(1,:)) + par(2)*sin(t);
% reshape back...
xdot = xdot(:);
end
```

Listing 1: Function 'pendulumODE' representing eq. (7)

The grid of initial conditions is initialized in listing 2.

[2] N.B. ode45 unfortunately does not support multiple column input / output for different initial conditions.

Florian Mahlknecht
fmahlknecht@student.ethz.ch
19-945-351

```
n = 100;
delta_x = 2*pi/(n-1);
x1_i = -pi:delta_x:pi;
x2_i = -pi:delta_x:pi;
[X0_1, X0_2] = meshgrid(x1_i,x2_i);

X0_shaped = grid2List(X0_1,X0_2);

% define the time window...
t0 = 0.0;
%dt = 50E-3;
t1 = 300;
tspan = [0 t1]; %0:dt:t1;
```

Listing 2: Matlab script 'initSpaceTimeGrid'

Crucial is hereby the function `grid2List`, presented in listing 3.

```
function [X0_shaped] = grid2List(X0_1,X0_2)
%GRID2LIST wraps meshgrid into single vector

X0 = [X0_1(:), X0_2(:)];
X0_shaped = X0';
X0_shaped = X0_shaped(:);
end
```

Listing 3: Matlab 'grid2List'

Results from `ode45` are then transformed back again with `list2Grid` given in listing 4.

```
function [X1_1,X1_2] = list2Grid(X1, X0_1, X0_2)
%LIST2GRID Maps long list vector back to array
%   Needs X0 to know dimensions (makes things easier here)

[m,n] = size(X1);
% make matrix with two long columns first to maintain couples of (x1,x2)
X1 = reshape(X1,2,n/2)';

%X1(:,1) = wrapToPi(X1(:,1));

% get grid sizes
[m,n] = size([X0_1, X0_2]);
[m,n1] = size(X0_1);

rshpd = reshape(X1,m,n);
X1_1 = rshpd(:,1:n1);
X1_2 = rshpd(:,(n1+1):n);
end
```

Listing 4: Matlab 'list2Grid'

Having a look at the main implementation yields details on the workflow, see listing 5.

fmahlknecht@student.ethz.ch
19-945-351

```
% Nonlinear Dynamics - FLTE Visualization

clear
%close all

k = 0.1;
a = 0.5;

par = [k a];
%par = [0 0]; % uncomment for part (i)

initSpaceTimeGrid;

[t,X1] = ode45(@(t,x) pendulumODE(t,x,par), tspan, X0_shaped);

[X1_1, X1_2] = list2Grid(X1(length(t),:), X0_1, X0_2);
FTLE = calcFTLE(X1_1, X1_2, x1_i, x2_i, t1, t0);
plotFTLE(X0_1, X0_2, FTLE, sprintf('k = %0.1f; a = %0.1f; t = %0.0f',k,a,t1));
```

Listing 5: Main script executing all functions

For calculating the FTLE, the already familiar implementation in listing 6 is shown. Note that the logarithmic scaling is added for a more granular comparison in the output plots.

```
function [FTLE] = calcFTLE(X1_1, X1_2, x1_i, x2_i, t1, t0)
%CALCFTLE Calculates the finite time liapunov exponent
%   x1_i and x2_i are used to calculate the differences in those directions
[DF1, DF2] = gradient(X1_1, x1_i, x2_i);
[DF3, DF4] = gradient(X1_2, x1_i, x2_i);

DF = [DF1,DF2;DF3,DF4];

C11=DF1.^2+DF3.^2; C12=DF1.*DF2+DF3.*DF4;
C21=C12; C22=DF2.^2+DF4.^2;

detC = C11.*C22-C12.*C21;
traceC = C11+C22;

FTLE = 1/(2*(t1-t0))*log(real(traceC/2+sqrt((traceC./2).^2-detC)));
end
```

Listing 6: Matlab 'calcFTLE'

Listing 7 shows the complete code used to print the figures. Even though it may not be adequate, a contour plot in combination with a color scale was used to print the surface in two dimensions.

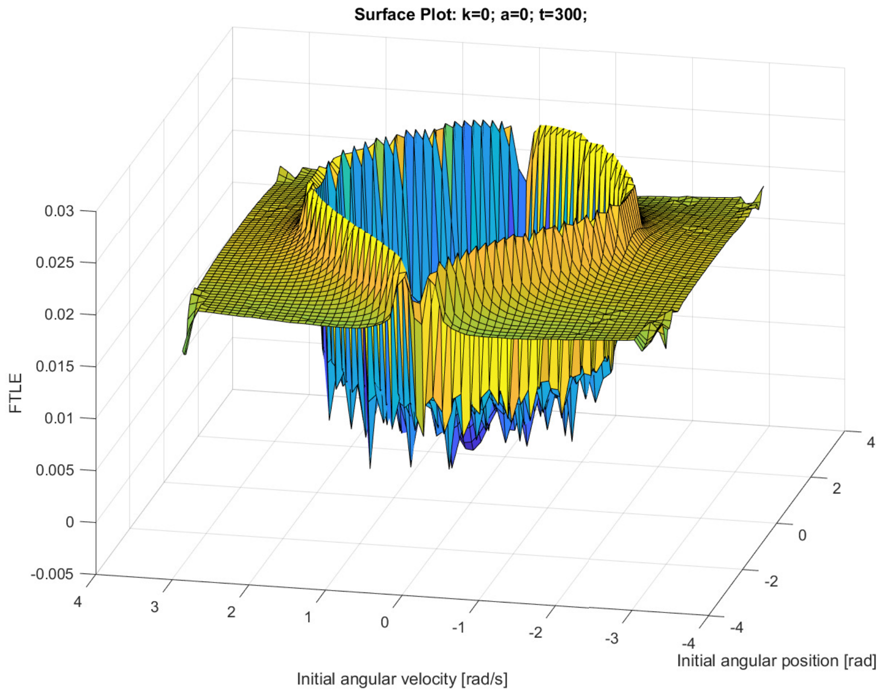
Florian Mahlknecht
fmahlknecht@student.ethz.ch
19-945-351

```
function plotFTLE(X0_1, X0_2, FTLE, info)
%PLOTFTLE Plots the FTLE for 2-dim initial condition grid adequately

%figure1 = figure('Name','Surface Plot');
%hold on;
%title(figure1.Name,'FontSize',16,'Interpreter','latex');
%axes1 = axes('Parent',figure1);
%hold(axes1,'on');
surf(X0_1, X0_2, FTLE);
xlabel("Initial angular position [rad]");
ylabel("Initial angular velocity [rad/s]");
zlabel("FTLE");
%xticks(0:pi/2:2*pi);
%xticklabels({'0','$\frac{\pi}{2}$','$\pi$','$3\,\frac{\pi}{2}$','$2\,\pi$'});

figure2 = figure('Name',strcat("Contour Plot: ", info));
axes2 = axes('Parent',figure2);
hold(axes2,'on');
contourf(X0_1, X0_2, FTLE)
colorbar
title(figure2.Name,'FontSize',16,'Interpreter','latex');
set(axes2,'FontSize',12,'TickLabelInterpreter','latex');
xticks(-2*pi:pi/2:2*pi);
xticklabels({'$-2\,\pi$', '$-3\,\frac{\pi}{2}$', ...
    '$-\pi$', '$-\frac{\pi}{2}$',...
    '0','$\frac{\pi}{2}$','$\pi$', ...
    '$3\,\frac{\pi}{2}$','$2\,\pi$'});

yticks(-2*pi:pi/2:2*pi);
yticklabels({'$-2\,\pi$', '$-3\,\frac{\pi}{2}$', ...
    '$-\pi$', '$-\frac{\pi}{2}$',...
    '0','$\frac{\pi}{2}$','$\pi$', ...
    '$3\,\frac{\pi}{2}$','$2\,\pi$'});
xlabel("Initial angular position [rad]");
ylabel("Initial angular velocity [rad/s]");

% SAVE
set(gcf, 'PaperUnits', 'centimeters');
x_width=15 ;y_width=10;
set(gcf, 'PaperPosition', [0 0 x_width y_width]); %
fileName = strcat('../latex/figures/', ...
    regexprep(strrep(lower(figure2.Name), ' ', '-'), ...
              '[\[\]:=]', ''), '.eps');
saveas(gcf, fileName,'epsc');
end
```

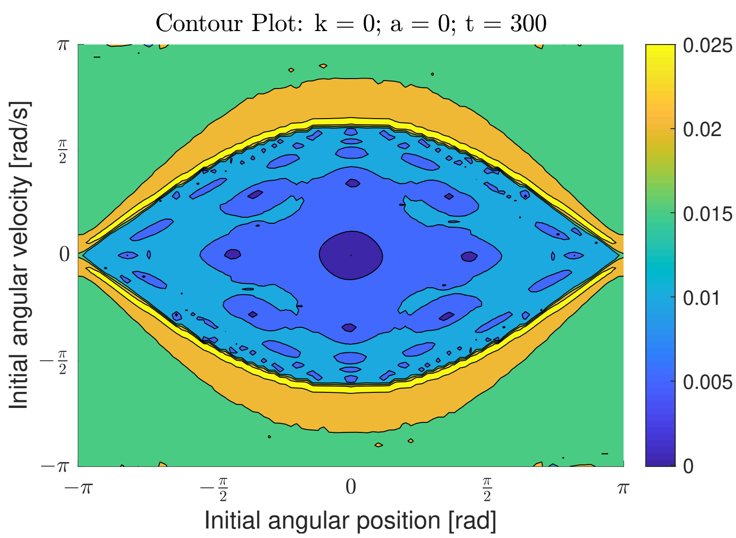
Listing 7: Matlab 'plotFTLE'

fmahlknecht@student.ethz.ch
19-945-351

3.2 Results



(a) Surface Plot



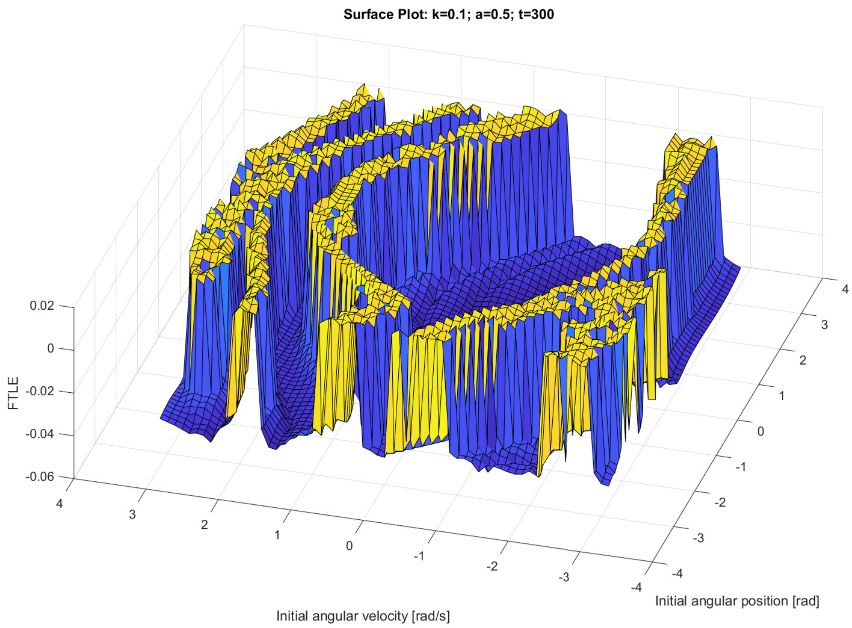
(b) Contour Plot

Figure 5: Undamped nonlinear pendulum FTLE

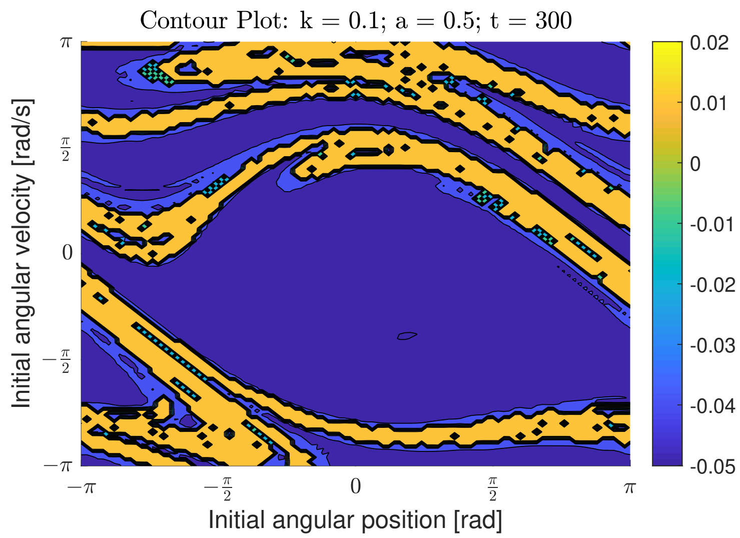
As expected the main separatrix with the highest ridge shows up as an ellipses dividing the inner region from the outer region. The yellow region with the highest sensitivity is indeed rather narrow and can clearly be identified as the separatrix. In the inner regions solutions simply oscillate around the stable fixpoint $\varphi = 0$ with an amplitude given from the initial condition (and since the pendulum is not damped, they endlessly do so). In the outer region solutions perform complete revolutions, because the speed limit is reached.

Florian Mahlknecht
fmahlknecht@student.ethz.ch
19-945-351

A closer look to fig. 5b reveals that the outside region with higher speed, is on a higher sensitivity level than the inner region, i.e. the finite time liapunov exponent is higher. However, we need to keep in mind that the plot uses the logarithmic scale, meaning that with respect to the separatrix both regions are insensitive to initial conditions in absolute comparison.



(a) Surface Plot

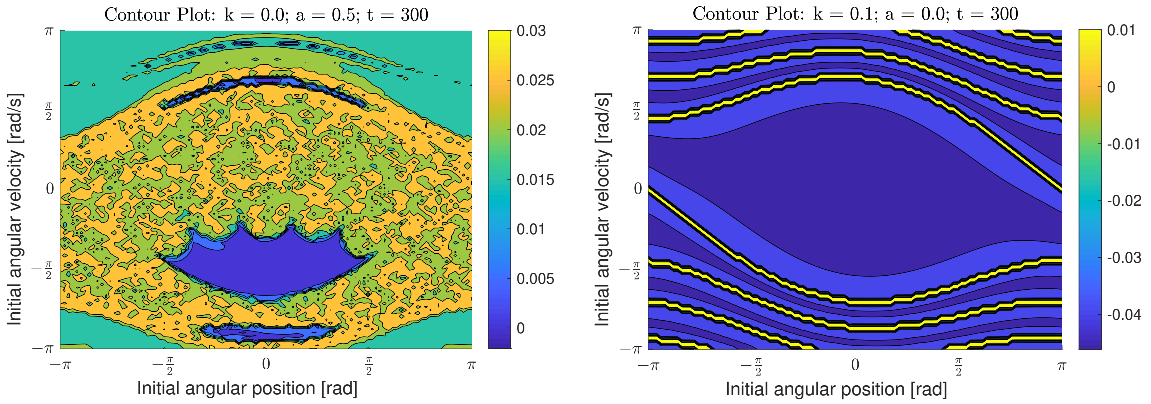


(b) Contour Plot

Figure 6: Damped and enforced nonlinear pendulum FTLE

Figure 6 shows the results for parameter values different from 0, namely $\begin{pmatrix}k & a\end{pmatrix} = \begin{pmatrix}0.1 & 0.5\end{pmatrix}$. As we can see, there are clear ridges visible with interesting geometric forms. To get better insights on how the parameters affect those, we generate two figures which separate damping and enforcement effects.

fmahlknecht@student.ethz.ch
19-945-351



(a) Enforcement only ($k = 0$) (b) Damping only ($a = 0$)

Figure 7: Separate damping and enforcement - FTLE Contour Plot

Figure 7 shows the two effects. As we can see the enforcement (fig. 7a) generates repulsive ridges which are symmetric with respect to the velocity axis, because there is no relation with position depending on parameter a in eq. (7) on page 6. Recalling the relevant part:

$$f_2(\mathbf{x}, t) = -k\,x_2 - \sin x_1 + a \sin t \tag{8}$$

The time dependency however, yields an asymmetric behavior in velocities. There seems to be an area with low sensitivity to initial conditions with negative initial velocities. A possible explanation therefor could be that starting from this region makes it easier to grasp the enforced frequency. In better detail we have that in those starting points x_1 is decreasing, approaching some angle near $-\pi$, also following from eq. (7) on page 6:

$$\dot{x}_1 = x_2 < 0 \iff x_2 < 0$$

By rearranging eq. (8), also neglecting the term $k = 0$, we may get:

$$f_2(\mathbf{x}, t) = +\sin(-x_1) + a \sin t$$

Given that x_1 is decreasing, as discussed before, $-x_1$ is increasing, just like time t. So they basically push in the same direction, which is why the solutions starting from that region $x_1 \approx 0$ and $x_2 < 0$ may be more likely to get a stable trajectory with the enforced frequency.

Figure 7b on the other hand, shows repeating separatrices in the velocity space. For a damped pendulum this is nothing but the count of how many revolutions it takes to approach the asymptotically stable fixpoint $\theta = 0$. Wherever we encounter a boundary velocity which in the last rotation comes close to the mechanically critical configuration $\theta = \pi$, there is a major separatrix ridge.

Moreover, there is an interesting asymmetric behavior around $-\pi$ and π respectively: if we approach $-\pi$ from the left side, meaning we use negative angular values to finally get to $-\pi$, the separatrices lie on the interesting values $\dot{\theta} = 0$ or some positive value $\dot{\theta} \approx \frac{\pi}{2}$. For the stable values in between the physical meaning is that even for some velocity values $\dot{\theta} > 0$ the damping is strong enough to approach the stable fixpoint $\theta = 0$ without reaching critical situations anymore. The interesting separatrix in $\dot{\theta} \approx \frac{\pi}{2}$, represents the first solutions which encounter the next critical

Florian Mahlknecht
fmahlknecht@student.ethz.ch
19-945-351

point on the completion of the first revolution. However, for small $\dot{\theta} < 0$ when approaching $-\pi$ from the left, there is no such separatrix, since solutions will converge back to $\theta = 0$ from the left side.

When approach $+\pi$ from the right side, the situation is similar, but now for *negative* values of angular velocities. So generally we might observe that the plot is point symmetric at 180°with respect to the origin.

By inspection, it is possible to appreciate the combination of the enforcement and damping effect, resulting in fig. 6b on page 11.

As a final figure, the contour plot is shown for backward time analysis (inverting the velocity axis, which carries the minus sign from backward time). In this way fig. 8 reveals more insight on the attractor geometries, which in fig. 6b are just low sensitivity regions.

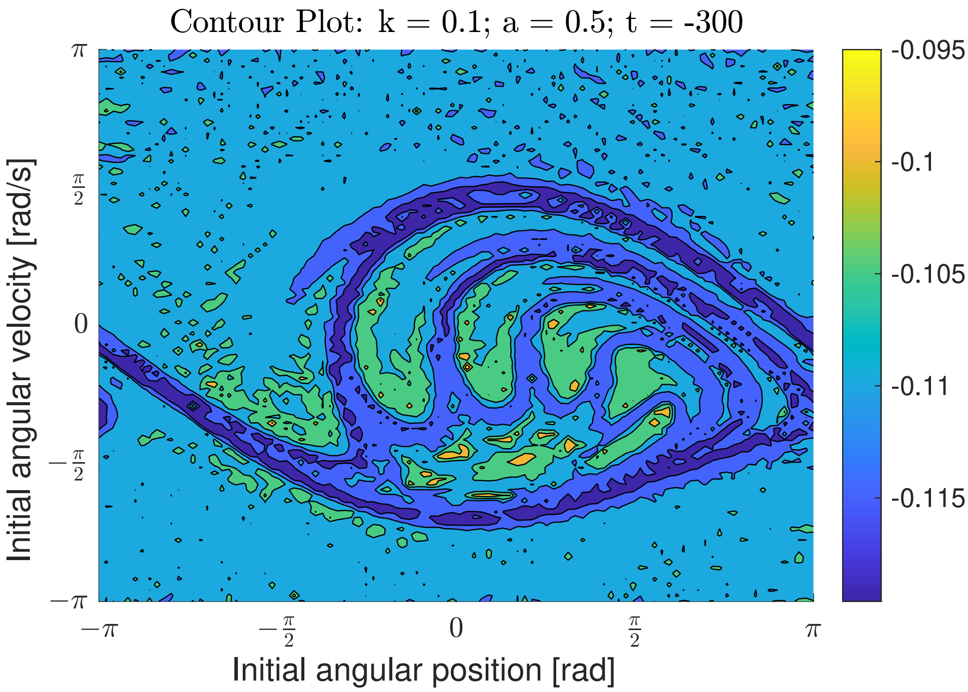


Figure 8: Contour plot backwards in time

According to the color scheme of fig. 8, the green and yellow regions are areas where solutions might be heading to.

fmahlknecht@student.ethz.ch
19-945-351